Modeling Sample and Hold

Digital Control System Analysis and Design.
C.L.Philips & H.T.Nagle. 3.ed.

Sampling and Reconstruction

3.1 INTRODUCTION

In Chapter 2 the concept of a discrete system was developed. We found that a discrete system is described (modeled) by a difference equation and that signals within the system are described by number sequences (e.g., $\{e(k)\}$). Some of these number sequences may be generated by sampling a continuous time signal (e.g., in digital control systems). To provide a basis for thoroughly understanding the operation of digital control systems, it is necessary to determine the effects of sampling a continuous-time signal. These topics are investigated in this chapter.

Sections 3.8 and 3.9 are devoted to the internal operation of digital-to-analog (D/A) and analog-to-digital (A/D) converters. A background in electrical engineering is needed to understand much of this material. However, the nonelectrical engineer will be able to understand the characteristics of different types of A/D and D/A converters by reading these sections.

3.2 SAMPLED-DATA CONTROL SYSTEMS

In this section the type of sampling that generally occurs in sampled-data control systems and in digital control systems is introduced, and a mathematical model of this sampling is developed. From this model we may determine the effects of the sampling on the information content of the signal that is sampled.

To introduce sampled-data systems, we consider the radar tracking system of Figure 3-1a. This system is described in Section 1.5. We consider only the control

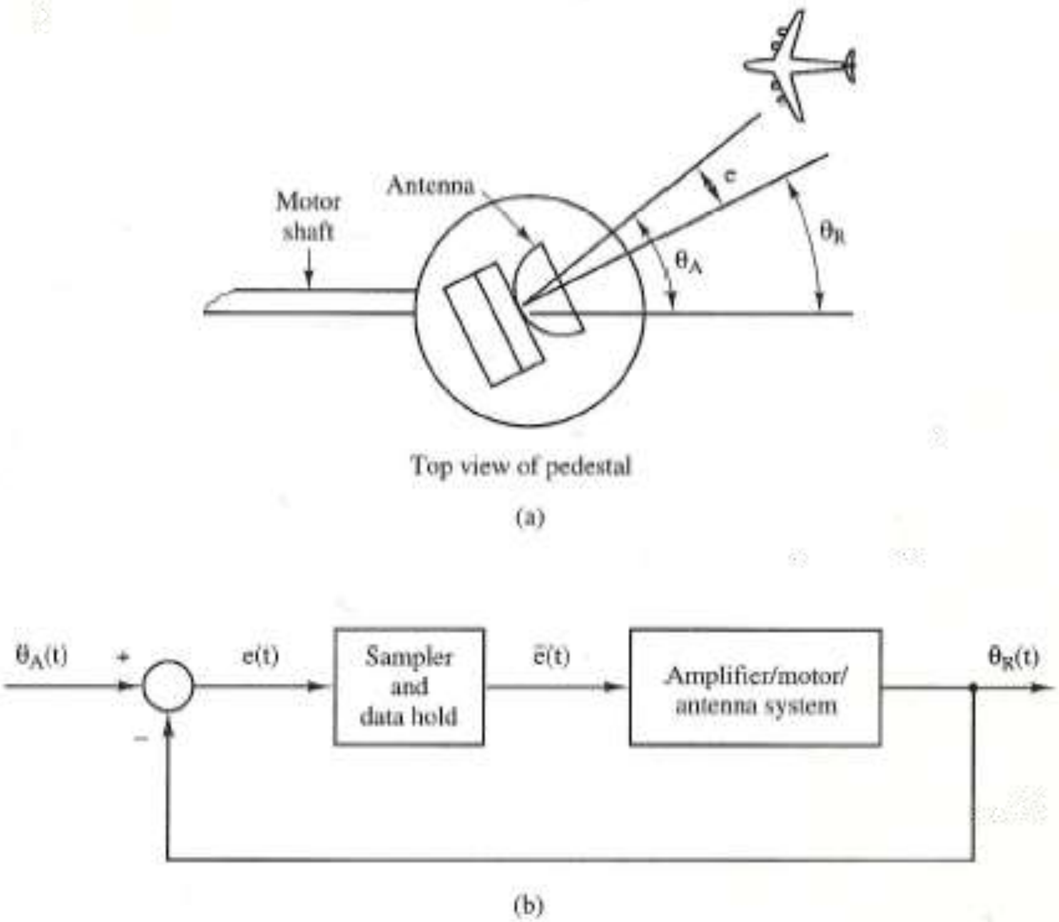


Figure 3-1 Sampled-data control system.

of the yaw angle $\theta_R(t)$, shown in the top view of the pedestal. The closed-loop system is to track the aircraft shown automatically. In Figure 3-1a, $\theta_R(t)$ is the yaw-axis point angle of the antenna, and $\theta_A(t)$ is the angle to the aircraft. Hence the tracking error is $e(t)$, given by

$$e(t) = \theta_A(t) - \theta_R(t)$$

Assume that the radar transmits every T seconds. Then the error $e(t)$ is known only every T seconds. The block diagram of this system is shown in Figure 3-1b. The radar receiver must output a voltage at every instant of time to the power amplifier. Since only $e(kT)$ is known, a decision must be made as to the form of power amplifier input, $\bar{e}(t)$, for $t \neq kT$.

In general, it is undesirable to apply a signal in sampled form, such as a train of narrow rectangular pulses, as shown in Figure 3-2, to a plant, because of the high-frequency components inherently present in that signal. Therefore, a data-reconstruction device, called a data hold, is inserted into the system directly following the sampler. The purpose of the data hold is to reconstruct the sampled signal into a form that closely resembles the signal before sampling. The simplest data-reconstruction device, and by far the most common one, is the zero-order hold. The

Husk:
I det følgende er e(t) et signal
є betegner eksponentialfunktionen

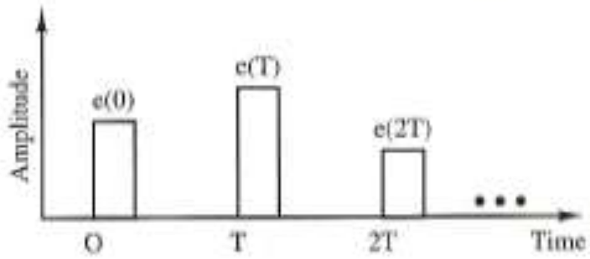


Figure 3-2 Sampled signal in pulse form.

operation of a sampler/zero-order hold combination is described by the signals shown in Figure 3-3. The zero-order hold clamps the output signal to a value equal to that of the input signal at the sampling instant.

The sampler and zero-order hold can be represented in block diagram form as shown in Figure 3-4. The signal $\bar{e}(t)$ can be expressed as

$$\begin{aligned}\bar{e}(t) = e(0)[u(t) - u(t-T)] + e(T)[u(t-T) - u(t-2T)] \\ + e(2T)[u(t-2T) - u(t-3T)] + \cdots\end{aligned} \tag{3-1}$$

where $u(t)$ is the unit-step function. The Laplace transform of $\bar{e}(t)$ is $\bar{E}(s)$, given by

$$\begin{aligned}\bar{E}(s) &= e(0)\left[\frac{1}{s} - \frac{\epsilon^{-Ts}}{s}\right] + e(T)\left[\frac{\epsilon^{-Ts}}{s} - \frac{\epsilon^{-2Ts}}{s}\right] \\ &\quad + e(2T)\left[\frac{\epsilon^{-2Ts}}{s} - \frac{\epsilon^{-3Ts}}{s}\right] + \cdots \\ &= \left[\frac{1-\epsilon^{-Ts}}{s}\right][e(0) + e(T)\epsilon^{-Ts} + e(2T)\epsilon^{-2Ts} + \cdots] \\ &= \left[\sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs}\right]\left[\frac{1-\epsilon^{-Ts}}{s}\right]\end{aligned} \tag{3-2}$$

(The Laplace transform is reviewed in Appendix VIII, and Appendix IX gives a table of Laplace transforms.)

The first factor in the last expression in (3-2) is seen to be a function of the input signal $e(t)$ and the sampling period T. The second factor is seen to be independent of $e(t)$ and therefore can be considered to be a transfer function. Thus the sampler/

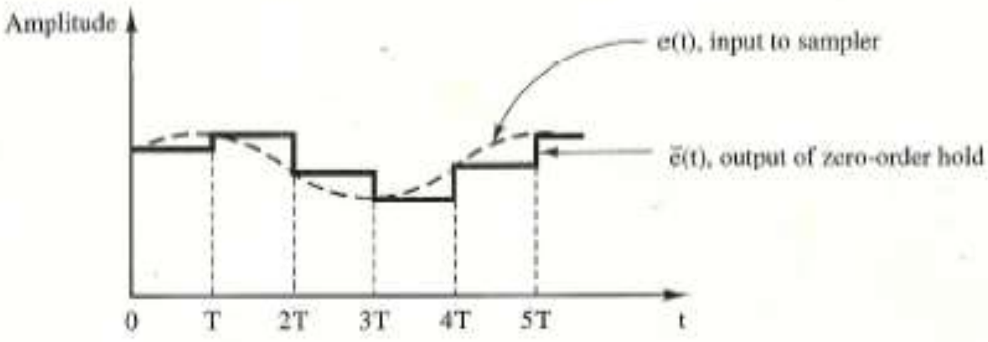


Figure 3-3 Input and output signals of sampler/data hold.

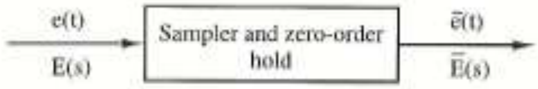


Figure 3-4 Sampler and data hold.

hold operation can be represented as shown in Figure 3-4. The function $E^*(s)$, called the starred transform, is *defined* as

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs} \tag{3-3}$$

Hence (3-2) is satisfied by the representation in Figure 3-5. The operation denoted by the switch in Figure 3-5 is defined by (3-3) and is called an ideal sampler; the operation denoted by the transfer function is called the data hold. It is to be emphasized that $E^*(s)$ *does not appear in the physical system* but appears as a result of factoring (3-2). The sampler (switch) in Figure 3-5 does not model a physical sampler and the block does not model a physical data hold. However, the combination does accurately model the input–output characteristics of the sampler-data hold device, as demonstrated earlier.

The operation symbolized by the sampler in Figure 3-5 cannot be represented by a transfer function. From (3-3) we see that the output of the sampler is a function of $e(t)$ only at $t = kT, k = 0, 1, 2, \ldots$. Hence many different input signals can result in the same output signal $E^*(s)$. However, the representation of a sampler as a transfer function would require each different $E(s)$ to result in a different $E^*(s)$. Hence no transfer function exists for the sampler, and this property of the sampler complicates the analysis of systems of the type shown in Figure 3-1b.

As an aside, many of the systems that we consider will have unity gain in the feedback path, as shown in Figure 3-1b. In practical systems, the sensor transfer function will appear in the feedback path. However, the bandwidth of the sensor is usually much greater than that of the plant, and thus the sensor can be considered to be a constant gain for system analysis and design. For a discussion of the problems in converting practical control systems to systems with unity gain in the feedback path, see Ref. 1.

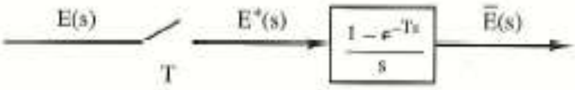


Figure 3-5 Representation of sampler and data hold.

3.3 THE IDEAL SAMPLER

The inverse Laplace transform of $E^*(s)$ is, from (3-3),

$$\begin{aligned} e^*(t) = \mathscr{L}^{-1}[E^*(s)] &= e(0)\delta(t) + e(T)\delta(t-T) \\ &+ e(2T)\delta(t-2T) + \cdots \end{aligned} \tag{3-4}$$

where $\delta(t)$ is the unit impulse function occurring at $t = 0$. Then $e^*(t)$ is a train of impulse functions whose weights are equal to the values of the signal at the instants of sampling. Thus $e^*(t)$ can be represented as shown in Figure 3-6, since the impulse function has infinite amplitude at the instant it occurs. Note again that $e^*(t)$ is not a physical signal.

The sampler that appears in a sampler/hold model is usually referred to as an *ideal sampler*, since nonphysical signals (impulse functions) appear on its output. This sampler is also referred to as an *impulse modulator*. To demonstrate this modulation concept, we define

$$\delta_T(t) = \sum_{n=0}^{\infty} \delta(t - nT) = \delta(t) + \delta(t - T) + \cdots \tag{3-5}$$

Then $e^*(t)$ can be expressed as

$$\begin{aligned} e^*(t) &= e(t)\delta_T(t) = e(t)\delta(t) + e(t)\delta(t - T) + \cdots \\ &= e(0)\delta(t) + e(T)\delta(t - T) + \cdots \end{aligned} \tag{3-6}$$

In this form it can be readily seen that $\delta_T(t)$ is the carrier in the modulation process, and $e(t)$ is the modulating signal [2]. The Laplace transform requires $e(t)$ to be zero for $t < 0$ [3]. For this reason the summation in (3-5) can be taken from $n = -\infty$ to $n = \infty$ with no change in (3-6). Two equivalent representations of the ideal sampler are given in Figure 3-7.

A problem arises in the definition of the ideal sampler output in (3-4) if $e(t)$ has a discontinuity at $t = kT$. For example, if $e(t)$ is a unit-step function, what value is used for $e(0)$ in (3-4)? In order to be consistent in the consideration of discontinuous signals, the output signal of an ideal sampler is *defined* as follows:

Definition. The output signal of an ideal sampler is defined as the signal whose Laplace transform is

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs} \tag{3-7}$$

where $e(t)$ is the input signal to the sampler. If $e(t)$ is discontinuous at $t = kT$, where k is an integer, then $e(kT)$ is taken to be $e(kT^+)$. The notation $e(kT^+)$ indicates the value of $e(t)$ as t approaches kT from the right (i.e., at $t = kT + \epsilon$, where ϵ is made arbitrarily small).

med $z = \epsilon^{Ts} = e^{Ts}$
ses ligning (3-7) netop at være definitionen på z-transformationen

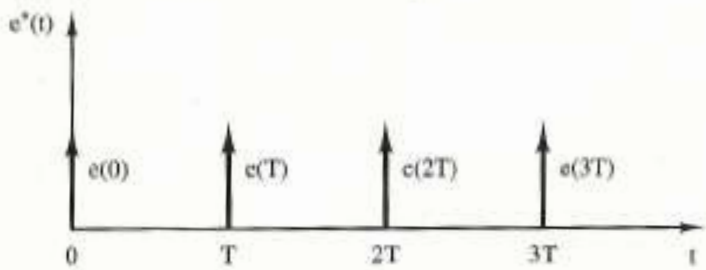


Figure 3-6 Representation of $e^*(t)$.

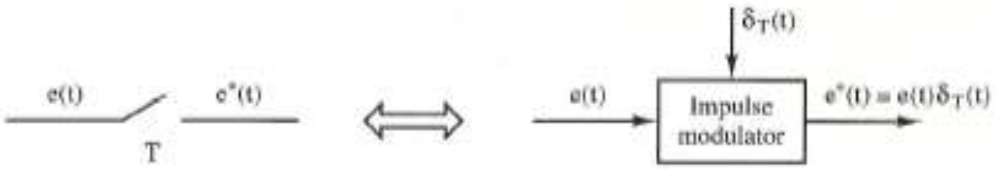


Figure 3-7 Representations of the ideal sampler.

It is important to note that the definition of the sampling operation as specified in (3-7) together with the zero-order-hold transfer function defined by

$$G_{ho}(s) = \frac{1 - \epsilon^{-Ts}}{s} \tag{3-8}$$

yield the correct mathematical description of the sampler/hold operation defined by (3-2). It should also be noted that if the signal to be sampled contains an impulse function at a sampling instant, the Laplace transform of the sampled signal does not exist; but since continuous signals in practical situations never contain impulse functions, this limitation is of no practical concern.

Example 3.1

Determine $E^*(s)$ for $e(t) = u(t)$, the unit step. For the unit step, $e(nT) = 1$, $n = 0, 1, 2, \ldots$. Thus from (3-7),

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\epsilon^{-nTs} = e(0) + e(T)\epsilon^{-Ts} + e(2T)\epsilon^{-2Ts} + \cdots$$

or

$$E^*(s) = 1 + \epsilon^{-Ts} + \epsilon^{-2Ts} + \cdots$$

$E^*(s)$ can be expressed in closed form using the following relationship. For $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

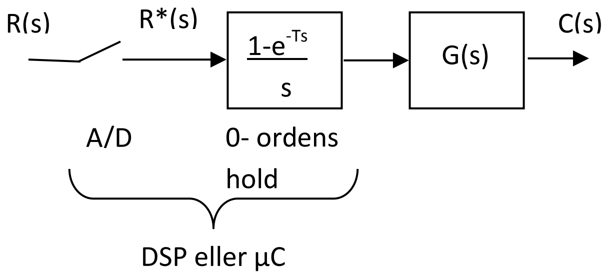
The condition $|x| < 1$ guarantees convergence of the series. Hence the expression for $E^*(s)$ above can be written in closed form as

$$E^*(s) = \frac{1}{1 - \epsilon^{-Ts}}, \qquad |\epsilon^{-Ts}| < 1$$

---

med $z = \epsilon^{Ts} = e^{Ts}$ fås netop $E(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$ **som kendt fra tidligere for et step**

Modellering af sample / hold har som konsekvens, at indsættes en µC eller DSP i signalvejen, ændres systemets model som i figuren nedenfor. Her tænkes der ikke i DSP'en eller µC'eren, at ske andet end at tallet ganges med én.



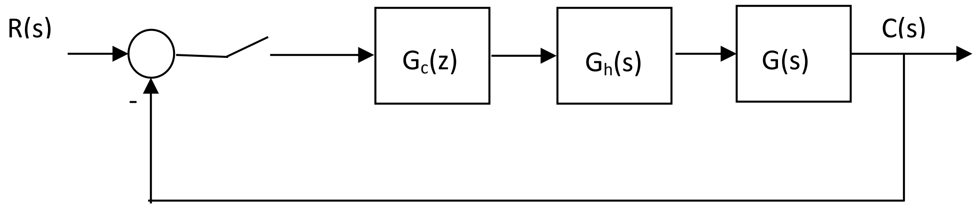
Der kan i praksis ikke opskrives et udtryk for C(s) som funktion af R(s) og sample / hold, da A/D-konverteringen ikke kan modelleres som en overføringsfunktion.
Opskrevet i s-planen er C(s)= G(s)·G_{hold}(s) ·R*(s), men udtrykket ville indeholde R*(s), der er en række med potenser af e^{-Ts}, hvilket gør udtrykket uegnet til beregning.
Derimod kan der opskrives udtryk i z-planet:

$$\mathbf{C(z) = Z\{\frac{1-e^{-Ts}}{s} \cdot G(s)\} \cdot R(z) = (1-z^{-1}) \cdot Z\{\frac{G(s)}{s}\} \cdot R(z)}$$

Bemærk , at (1-**e-**$^{\mathbf{Ts}}$**)** erstattes med (1-z^{-1}), hvorimod s'et skal z-transformeres sammen med G(s). Herved kommer virkningen af holdenetværket med, herunder den forsinkelse på et halvt sampleinterval, der tillige er konsekvensen. I lukket-sløjfen er det negative fasebidrag fra forsinkelsen afgørende for valg af samplefrekvens og/eller valg af systemets båndbredde ≈ $\omega_{\Phi m}$

I den generelle situation indeholder µC'en eller DSP'en det program, der realiserer den kompensation/regulator vi har beregnet som G_c(s). Senere ser vi på hvordan et givet G_c(s) kan omsættes til G_c(z) og dermed programmeres.

Dvs. den samlede lukkede sløjfe kan modelleres som vist på figuren nedenfor:



Der kan ikke dannes et udtryk for C(s)/R(s), men for C(z)/R(z), dvs. vi får kun sammenhørende værdier for indgangssignalet r(t) og udgangssignalet c(t) til sampletidspunkterne, repræsenteret som r(nT) og c(nT).

$\frac{C(z)}{R(z)} = \frac{G_c(z) \cdot \overline{G_h G(z)}}{1 + G_c(z) \cdot \overline{G_h G(z)}}$, hvor stregen over $G_hG(z)$ betyder Z-transformation samlet som vist ovenfor.

Eksempel:

Vi påvirker et 1.ordenssystem $\mathbf{G(s)} = \dfrac{1}{\mathbf{s+1}}$ med et enhedsstep. Resultatet heraf vil i det analoge tilfælde

være: $C(s) = \dfrac{1}{s+1} \cdot \dfrac{1}{s} = \dfrac{-1}{s+1} + \dfrac{1}{s} \quad \Rightarrow \quad c(t) = (1 - e^{-t}) \cdot u(t)$

Med sample / hold i signalvejen fås:

$$\mathbf{C(z)} = (1 - \mathbf{z}^{-1}) \cdot Z\{\frac{1}{\mathbf{s(s+1)}}\} \cdot \mathbf{R(z)} \quad \textbf{Hvor } \mathbf{R(z)} = \frac{\mathbf{z}}{\mathbf{z}-1} \textbf{ for et enhedsstep}$$

$$= (1 - \mathbf{z}^{-1}) \cdot \left[\frac{\mathbf{z}}{\mathbf{z}-1} + \frac{-\mathbf{z}}{\mathbf{z}-\mathbf{e}^{-\mathbf{T}}}\right] \cdot \frac{\mathbf{z}}{\mathbf{z}-1} = \left[\frac{1-\mathbf{e}^{-\mathbf{T}}}{\mathbf{z}-\mathbf{e}^{-\mathbf{T}}}\right] \cdot \frac{\mathbf{z}}{\mathbf{z}-1} \quad \Rightarrow \quad \frac{\mathbf{C(z)}}{\mathbf{z}} = \left[\frac{1}{\mathbf{z}-1} + \frac{-1}{\mathbf{z}-\mathbf{e}^{-\mathbf{T}}}\right]$$

$\mathbf{C(z)} = \left[\dfrac{\mathbf{z}}{\mathbf{z}-1} + \dfrac{-\mathbf{z}}{\mathbf{z}-\mathbf{e}^{-\mathbf{T}}}\right] \quad \Rightarrow \quad \mathbf{c(nT)} = \mathbf{1 - e}^{\mathbf{-nT}}$, der til sampletidspunkterne har samme værdi som i det analoge tilfælde. Modellen sikrer altså at DC-forstærkningen er korrekt.

Havde vi i stedet blot taget C(z) = G(z) ·R(z) var DC-forstærkningen = $(1-e^{-T})^{-1}$ dvs. afhængig af sample intervallet og altid større end 1.