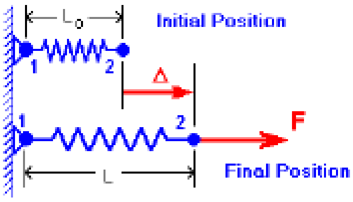
## 2.2 A Simple Elastic Spring

A simple example of a discrete element is an elastic spring, as shown in the figure below:

**Figure 1: One Elastic Spring**




The left end of the spring is rigidly connected to a wall, while the other end is free to move only in a horizontal direction. A force, F, is applied at the free end. Using Hooke's law, a direct relationship between the axial load, F, in the spring and the deflection, Δ, at its free end is developed:

$F := 20 \cdot kN$ $\qquad$ $k := 1500 \cdot kN \cdot m^{-1}$

(1) $\qquad \Delta := \frac{1}{k} \cdot F$ $\qquad$ $\Delta = 0.013\,m$

where k is the spring stiffness or spring constant.

Isolating F of equation (1) results in the force-deflection relationship for the spring:

(2) $\qquad F := k \cdot \Delta$ $\qquad$ $F = 2 \times 10^{4}\,N$

The stiffness, k, can be interpreted as the force required to produce a unit deflection (Δ= 1).

Observe that for this very simple example, the free end displacement of the spring, Δ, can be calculated directly using equation (2). No discretization is required.

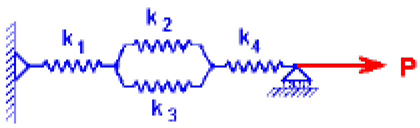
The next section presents a system of springs.

Units: $\qquad kN \equiv 1000 \cdot newton$ $\qquad$ $N \equiv newton$

## 2.3 A System of Springs

Suppose a system of springs was formed with each spring having a stiffness value, and a force was applied at the end as shown below:

**Figure 1: System of Four Springs**



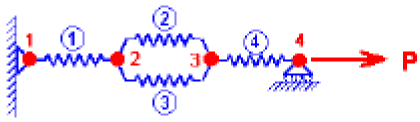

The following problem is stated:

For a given force, P, applied at the free end, determine the corresponding end displacement.

## 2.3.1 Step 1: Discretize the Spring System

For a discrete system, the discretization is well-defined since the properties of an individual spring are well-known. Thus, making one element equal to one spring, the finite element model of the system comprises four elements and four nodes, as shown below:

**Figure 1: Finite Element Model of Spring System**




The node labels of Figure 1are referred to as the **global** node numbering scheme. Each element has a local node numbering scheme which is used to formulate the element properties.

**Figure 2: Local Node Numbering Scheme**

## 2.3.2 Step 2: Select Interpolation Functions

In the finite element method, the interpolation across the element represents the variation of the field variable, which in the spring system is displacement. Recalling equation (2) of Section 2.2:

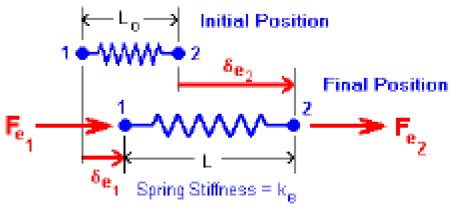
(1 - Section 2.2) $$F = k \cdot \Delta$$

Since the displacement varies linearly across the element, is **exact**, and each spring experiences a constant force, **no element interpolation is required**.

## 2.3.3 Step 3: Find the Element Properties

To determine the force-deflection characteristics of an element, consider a free-body diagram of a spring in its most general form of displacement:

**Figure 1: Free-body Diagram of Spring**




$L_o := 1 \cdot m$

$L := 2 \cdot m$

$k_e := 1500 \cdot kN \cdot m^{-1}$

$\delta_{e_2} := 1.25 \cdot m$

$\delta_{e_1} := .25 \cdot m$

$F := 20 \cdot kN$

The initial and final lengths of the springs are $L_o$ and L, respectively. If the deflection at the nodes is given by $\delta_{e1}$ and $\delta_{e2}$, then the net deflection, $\Delta$, is

(1a) $\Delta := L - L_o$ $\Delta = 1\,m$

or,

(1b) $\Delta := \delta_{e_2} - \delta_{e_1}$ $\Delta = 1\,m$

Hooke's law for a spring yields

(2a) $F_{e_2} := k_e \cdot \Delta$ or,

(2b) $F_{e_2} := k_e \cdot \left(\delta_{e_2} - \delta_{e_1}\right)$ $F_{e_2} = 1.5 \times 10^3\,kN$

where $k_e$ is the spring stiffness.

Applying equilibrium of forces ($\Sigma F_x = 0$),

(3) $F_{e_1} + F_{e_2} = 0$

Note that all forces and displacements, by convention, are positive to the right. Furthermore, the subscript "e" indicates that the quantities are referenced to a single element labeled "e".

The element equations for the spring can thus be formed by combining (2b) and (3). In matrix form,

**Element Equation**

(4a) $$K_e \cdot \delta_e = f_e$$

where

(4b) $$K_e := \begin{pmatrix} k_e & -k_e \\ -k_e & k_e \end{pmatrix} \qquad \delta_e := \begin{pmatrix} \delta_{e_1} \\ \delta_{e_2} \end{pmatrix} \qquad f_e := \begin{pmatrix} F_{e_1} \\ F_{e_2} \end{pmatrix}$$

The matrix $\mathbf{K_e}$ is referred to as the **element stiffness matrix**, and its elements are called **stiffness coefficients**. For a given nodal displacement vector, $d_e$, the nodal force vector, $\mathbf{f_e}$, can be calculated using equation (4a):

$$f_e := K_e \cdot \delta_e \qquad f_e = \begin{pmatrix} -1.5\times 10^3 \\ 1.5\times 10^3 \end{pmatrix} kN$$

**Properties of the Element Stiffness Matrix**

Observe two important properties of the stiffness matrix: it is **symmetric** and **does not have an inverse**. The latter property implies that the determinant is zero.

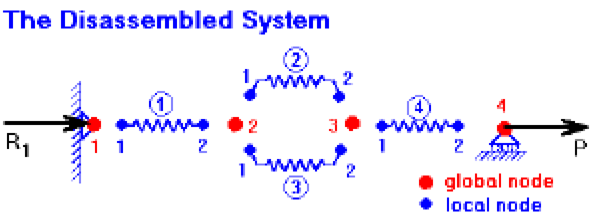
$$K_e = \begin{pmatrix} 1.5\times 10^3 & -1.5\times 10^3 \\ -1.5\times 10^3 & 1.5\times 10^3 \end{pmatrix} \frac{kN}{m} \qquad |K_e| = 0 \left(\frac{kN}{m}\right)^2$$

The **singularity** of the stiffness matrix prevents inverting equation (4a) to solve for the displacement vector $d_e$, given the force vector, $\mathbf{f_e}$. Physically, this means that a single spring cannot stretch unless one of its ends is fixed, which makes sense! To eliminate the problem, boundary conditions will be applied to the system of equations. However, before doing this, the system of springs must be assembled.

## 2.3.4 Step 4: Assemble the Elements

In the previous step, the force-displacement characteristics were considered for a single element. Here, the assembly procedure will express the force-displacement equations for the entire spring system. The figure below represents the disassembled spring system. Each element is assigned local node numbers (shown in blue) corresponding to the element equations of the previous section. The global nodes (shown in red) represent the discretized system when fully assembled.

**Figure 1:**




Red
Blue

**The System Topology**

From the figure above, a relationship exists between the local and global node numbers. Referred to as the **system's topology**, this relationship is best expressed in tabular form as shown at right:

| Element | Node Local | Node Global |
|---|---|---|
| ① | 1<br>2 | 1<br>2 |
| ② | 1<br>2 | 2<br>3 |
| ③ | 1<br>2 | 2<br>3 |
| ④ | 1<br>2 | 3<br>4 |

Table 1: The System Topology

The assembly procedure can be divided into two parts: compatibility and force equilibrium.

**1) Compatibility**

The first part expresses each element matrix equation in global form using the topology table. It is based on the compatibility rule, restated as:

**At a node, the value of the unknown nodal displacement must be the same for all elements connecting at that node.**

Recall the element equations for an element labeled e,

$$\begin{pmatrix} k_{e.} & -k_e \\ -k_e & k_e \end{pmatrix} \cdot \begin{pmatrix} \delta_{e_1} \\ \delta_{e_2} \end{pmatrix} = \begin{pmatrix} F_{e_1} \\ F_{e_2} \end{pmatrix}$$

to express each element equation using global node numbers:

(1a) $$\begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \cdot \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}^{\langle 1 \rangle}$$ ① 1 → 1, 2 → 2

(1b) $$\begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \cdot \begin{pmatrix} \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}^{\langle 2 \rangle}$$ ② 1 → 2, 2 → 3

(1c) $$\begin{pmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{pmatrix} \cdot \begin{pmatrix} \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}^{\langle 3 \rangle}$$ ③ 1 → 2, 2 → 3

(1d) $$\begin{pmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{pmatrix} \cdot \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} F_3 \\ F_4 \end{pmatrix}^{\langle 4 \rangle}$$ ④ 1 → 3, 2 → 4

The superscript on the force vector in the above equations distinguishes the force contribution from each element applied at a particular node. For example, in the second element, (1b), the force in element 2 is applied to nodes 2 and 3. In the third element, (1c), the force is applied to nodes 2 and 3.

**2) Equilibrium**

When the elements are assembled, local equilibrium is imposed at each node. In other words, the sum of all internal forces applied at a given node, i, must be equal to the resultant external load, $R_i$.

(2) $$\sum_e (F_i)^{\langle e \rangle} = (F_i)^{\langle 1 \rangle} + (F_i)^{\langle 2 \rangle} + (F_i)^{\langle 3 \rangle} + (F_i)^{\langle 4 \rangle} = R_i$$

Therefore, at node 1,

(3a) $(F_1)^{\langle 1 \rangle} = R_1$ ($R_1$ is the unknown reaction force at node 1.)

At node 2,

(3b) $(F_2)^{\langle 1 \rangle} + (F_2)^{\langle 2 \rangle} + (F_2)^{\langle 3 \rangle} = R_2 = 0$

At node 3,

(3c) $(F_3)^{\langle 2 \rangle} + (F_3)^{\langle 3 \rangle} + (F_3)^{\langle 4 \rangle} = R_3 = 0$

At node 4,

(3d) $(F_4)^{\langle 4 \rangle} = R_4 = P$ (P is the applied force at node 4.)

(Note that the superscript <i> refers to the force from element i, while the subscript k refers to the node k.)

**The Assembled Matrix Equation**

Combining equations (1a-d) and (3a-d) to eliminate the element forces yields the following matrix equation:

(4a) $K \cdot \delta = f$

$$k \equiv \begin{pmatrix} 1500 \\ 2000 \\ 2200 \\ 1200 \end{pmatrix}$$

where

(4b)
$$K := \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ 0 & (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \cdot \frac{kN}{m}$$

and

$$\delta = (\delta_1 \ \ \delta_2 \ \ \delta_3 \ \ \delta_4)^T \qquad f = (R_1 \ \ 0 \ \ 0 \ \ P)^T$$

The above system of equations (4a) represents the assembled force-displacement characteristics for the complete model. The matrix, **K**, is referred to as the **assembled stiffness matrix**. **K** has the same properties as the element stiffness matrix, $\mathbf{K_e}$; It is **symmetric** and **singular** because the determinant of **K** is zero (implying that it can not be inverted).

$$|K| = 0\,kg^{4}\,sec^{-8}$$

Viewing this property from another perspective, there are 5 unknowns ($d$ and $R_1$) in equation (4a), but only 4 equations. At least one node must be fixed to the ground so that the system will stretch under the applied load. The process of fixing a node is referred to as applying boundary conditions.

## 2.3.5

## Step 5: Apply the Boundary Conditions

Recall the assembled system of equations from the previous section:

(1a) $$K \cdot \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \\ 0 \\ P \end{pmatrix}$$

where

(1b) $$K := \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ 0 & (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \cdot \frac{kN}{m}$$

According to the model, node 1 is fixed, such that:

(2) $$\delta_1 = 0$$

The row and column of equation (1a), corresponding to node 1, can be eliminated for the time being:

(3) $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ 0 & (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ P \end{pmatrix}$$

The 3x3 non-zero submatrix of (3) can be considered separately

(4a) $$K_{mod} \cdot \begin{pmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$$

where $\mathbf{K_{mod}}$ is the modified stiffness matrix.

(4b)

$$K_{mod} := \begin{bmatrix} (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & -k_4 & k_4 \end{bmatrix} \cdot \frac{kN}{m}$$

Notice that $\mathbf{K_{mod}}$ has a non-zero determinant and can be inverted:

Assign values for the spring constants below:

$$k \equiv \begin{pmatrix} 1500 \\ 2000 \\ 2200 \\ 1200 \end{pmatrix} \frac{kN}{m}$$

$$|K_{mod}| = 7.56 \times 10^9 \left(\frac{kN}{m}\right)^3$$

$$K_{mod}^{-1} = \begin{pmatrix} 6.667 \times 10^{-4} & 6.667 \times 10^{-4} & 6.667 \times 10^{-4} \\ 6.667 \times 10^{-4} & 9.048 \times 10^{-4} & 9.048 \times 10^{-4} \\ 6.667 \times 10^{-4} & 9.048 \times 10^{-4} & 1.738 \times 10^{-3} \end{pmatrix} \frac{m}{kN}$$

This important property now allows one to calculate the nodal displacements for a given applied force vector. Details of the calculation are discussed in the next section.

## 2.3.6

## Step 6: Solve the System of Equations

After the boundary conditions have been applied to a system, the system of equations can be expressed as

$P \equiv 100 \quad kN$

$$k \equiv \begin{pmatrix} 1500 \\ 2000 \\ 2200 \\ 1200 \end{pmatrix} \frac{kN}{m}$$

(1a)

$$\begin{pmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = K_{mod}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$$

where

(1b)

$$K_{mod} := \begin{bmatrix} (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & -k_4 & k_4 \end{bmatrix} \cdot \frac{kN}{m}$$

and $\delta_1 := 0$

As mentioned in the previous section, $\mathbf{K_{mod}}$ is no longer singular. Thus the displacements can be solved using (1a). Since there are only three simultaneous equations for the example, the solution of (1a) will be solved symbolically. The symbolic inversion of $\mathbf{K_{mod}}$ yields

**Symbolic Inversion of $\mathbf{K_{mod}}$**

(2)

$$K_{mod}^{-1} = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{(k_1 + k_2 + k_3)}{[k_1 \cdot (k_2 + k_3)]} & \frac{(k_1 + k_2 + k_3)}{[k_1 \cdot (k_2 + k_3)]} \\ \frac{1}{k_1} & \frac{(k_1 + k_2 + k_3)}{[k_1 \cdot (k_2 + k_3)]} & \frac{(k_1 \cdot k_2 + k_1 \cdot k_3 + k_1 \cdot k_4 + k_4 \cdot k_2 + k_4 \cdot k_3)}{[k_1 \cdot [k_4 \cdot (k_2 + k_3)]]} \end{bmatrix}$$

Substituting (2) into (1a) produces the expression for the nodal displacements:

**Symbolic Solution**

(3)

$$\begin{pmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} := \begin{bmatrix} \frac{1}{k_1} \cdot P \\ \frac{(k_1 + k_2 + k_3)}{[k_1 \cdot (k_2 + k_3)]} \cdot P \\ \frac{(k_1 \cdot k_2 + k_1 \cdot k_3 + k_1 \cdot k_4 + k_4 \cdot k_2 + k_4 \cdot k_3)}{[k_1 \cdot [k_4 \cdot (k_2 + k_3)]]} \cdot P \end{bmatrix} \cdot m$$

$$\begin{pmatrix} \delta_2 & \delta_3 & \delta_4 \end{pmatrix} = \begin{pmatrix} 0.06667 & 0.09048 & 0.17381 \end{pmatrix} m$$

Alternatively, equation (1a) could be solved numerically to yield the same result:

**Numerical Solution**

$$\begin{pmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} := K_{mod}^{-1} \cdot \begin{pmatrix} 0 \cdot kN \\ 0 \cdot kN \\ P \cdot kN \end{pmatrix} \qquad \begin{pmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0.06667 \\ 0.09048 \\ 0.17381 \end{pmatrix} m$$

Once the unknown displacements have been determined, the magnitude of the reaction force, $R_1$, is computed using the original assembled matrix equation (prior to the application of boundary conditions):

**The Assembled Matrix Equation**

(4)

$$f := \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ 0 & (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \cdot \frac{kN}{m} \cdot \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix}$$

Numerically, the force vector is evaluated as:

$$f^T = \begin{pmatrix} -100 & 4.164 \times 10^{-14} & -5.828 \times 10^{-14} & 100 \end{pmatrix} kN$$

Symbolically, the force vector is computed using (3) and (4):

(5)

$$R_1 = -P$$

Note that the reaction force, $R_1$, at node 1 is equal and opposite to the applied force, P, thus satisfying global equilibrium.

The discrete system of springs could easily have been solved using a net stiffness derived from addition rules of network components in parallel and series configurations.