Chapter 3

Matrices

3.1 Keywords and phrases

Special matrices: (row vector, column vector, zero, square, diagonal, scalar, identity, lower triangular, upper triangular, symmetric, row echelon form, reduced row echelon form, augmented matrix); matrix arithmetic and operations:(equality of matrices, addition, subtraction, multiplication by scalar, matrix multiplication, pre-multiply, post-multiply, conformable for multiplication); terms or elements or entries of a matrix, size, pivot, transpose, inverse, invertible, singular, elementary row operations, row reduction, systems of linear equations, matrix representation of linear equations, linear span of the rows (or columns) of a matrix, rank, determinant, minors; finding inverses, the rank of a matrix, and solving simultaneous equations using elementary row operations, the relationship between the invertibility, rank, and determinant of a matrix, null space, homogeneous equation and solutions, particular solution.

3.2 Introduction

Matrices are sets of numbers arranged in rectangular arrays, and so can be viewed as a generalization of vectors. They provide a means of storing large quantities of information in such a way that each piece is easily identified and manipulated. But with matrices we can also deal with arrays of many numbers as single objects, denoting them by single symbols and performing calculations in elegant and compact ways. This is most notable when they are used to solve large systems of linear equations in ways which are easily programmed for computer implementation, and this is what we focus on in this chapter. In this context, they have been used in applications as diverse as electrical networks, mechanics, curve fitting in statistics and transportation problems.

To see how this works, consider the following system of linear equations:

$$\begin{array}{rcl} x+y &=& 6 \\ 3x+2y &=& 14 \end{array}.$$

As we shall see, it can be written as a matrix equation

$$AX = B,$$

where $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ is the *coefficient matrix* for the system, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is the *column matrix (vector)* of unknowns, and $B = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$ is the *forcing column matrix*. The term AX is the *matrix product* of A and X. If A and B were just numbers and $A \neq 0$, then we could solve for X by writing $X = A^{-1}B$. Under certain circumstances (namely when A is *invertible*) we can do the same thing in the more general situations. But even if A fails to have an inverse, it gives some information about *when* there are solutions of the system and *how many* there are. It is also possible to manipulate the matrix A (whether it is invertible or not) using *row reduction* to make it more obvious what the solutions are, when there are any. Before discussing these ideas, we briefly review basic matrix algebra.

3.3 Review of matrix algebra

3.3.1 Definitions

A matrix is a rectangular array of numbers (or more general objects like functions) such as

$$A=\begin{pmatrix}2&4\\3&6\\4&0\end{pmatrix},B=\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix},T=\begin{pmatrix}t_{11}&t_{12}&t_{13}\\t_{21}&t_{22}&t_{23}\\t_{31}&t_{32}&t_{33}\end{pmatrix}\text{ and }\Phi=\begin{pmatrix}\phi_{11}(x)&\phi_{12}(x)\\\phi_{21}(x)&\phi_{22}(x)\\\phi_{31}(x)&\phi_{32}(x)\end{pmatrix}$$

The elements in a matrix, for example, t_{31} in T above, are called the **terms**, **elements**, or **entries** of the matrix. They are identified by their subscripts, the first giving the row and the second the column in which the entry lies. Thus t_{ij} lies in the i^{th} row and the j^{th} column. It is often convenient to use the compact notation $[t_{ij}]$ for such a matrix T.

The **size** of a matrix is its number of rows and columns. Thus an $n\times m$ matrix has n rows and m columns. So T is a 3×3 matrix, whereas A and Φ are 3×2 matrices. We will mainly consider matrices which have 2 or 3 rows and columns. However, much of what we will do is easily extended to larger matrices, and at times we will consider larger examples.

3.3.2 Special matrices

There are various types of matrices which occur regularly in practice. For this reason they have been given special names.

1. A **row vector** is a matrix with just one row and a **column vector** is a matrix with a single column. For each of these, we only require one index to specify the position of an entry.

2. A **zero matrix** is one whose entries are all equal to 0. If Z is a zero matrix and A is any matrix of the same size, then $A+Z=A$. Where there is no confusion about size we shall simply write $Z=0$.

3. A **square matrix** is one which has the same number of rows and columns. Thus every $m\times m$ matrix is a square matrix. The matrix T above is a square matrix.

4. A **diagonal matrix** is a square matrix in which all entries are zero, except possibly those lying on the main diagonal; that is, those with subscripts i and j which are equal. (In fact, for *any* square matrix, these entries are called the **diagonal** entries.) Thus a diagonal matrix has the form

$$\begin{pmatrix}d_1&0&\cdots&\cdots&0&0\\0&d_2&&&&0\\\vdots&&\ddots&&&\vdots\\\vdots&&&\ddots&&\vdots\\0&&&&d_{n-1}&0\\0&0&\cdots&\cdots&0&d_n\end{pmatrix}$$

   where $d_1,d_2,\ldots,d_n$ are the diagonal entries.

   To describe a diagonal matrix, we often use a shorter form of notation in which only the diagonal elements are shown. The expression $\text{diag}(d_1,d_2,\ldots.,d_n)$ is another way of representing the diagonal matrix shown above.

5. A **scalar matrix** is a diagonal matrix whose diagonal elements are all equal. If the matrix $S=\text{diag}(\lambda,\lambda,\ldots,\lambda)$ is a scalar matrix, and if A is any other matrix of the same size, then the products satisfy $SA=AS=\lambda A$. It is possible to show that the converse is also true: if S is a square matrix which commutes with all other matrices of the same size, then S must be a scalar matrix.

6. An **identity** or **unit matrix** is a scalar matrix whose diagonal entries are all equal to 1. The $n\times n$ identity matrix $\text{diag}(1,1,\ldots,1)$ is denoted by I_n and is called the identity or unit matrix of size n. For any $n\times n$ matrix A, $I_nA=A=AI_n$, and for any $m\times n$ matrix A, $I_mA=A=AI_n$. We will often ignore the subscript n, and write I for the identity matrix when its size is obvious.

7. A **triangular matrix** is a square matrix whose entries on one side of the main diagonal are all 0. We say that $A = [a_{ij}]$ is **upper triangular** if and only if $a_{ij} = 0$ whenever $i > j$; that is, if and only if all entries below the main diagonal are 0. Thus upper triangular matrices have the following general form.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & \cdots & \cdots & 0 & a_{nn} \end{pmatrix}$$

   Similarly, A is **lower triangular** if and only if $a_{ij} = 0$ whenever $i < j$; that is, if and only if all entries above the diagonal are 0.

8. **Row echelon matrices** are important in the theory of linear equations. They are easier to recognize by sight than by their formal definition. However, in order to be row echelon a matrix must have the following properties:

   - The zero rows (that is, those containing nothing but 0) must appear below the non-zero rows.
   - The elements below any pivot are all 0. (We call the first non-zero element in any (non-zero) row a **pivot**.)

9. A **reduced row echelon matrix** is a row echelon matrix which satisfies the following additional properties:

   - Each pivot element is 1.
   - The elements above (and below) any pivot are all 0.

Exercise 39 *Decide which of the matrices A, B, C, and D below fall into any of the special types of matrices listed above?*

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \textit{ and } D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.3.3 Equality of matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they have the same size, and each element of one matrix is equal to the corresponding element of the other, that is $a_{ij} = b_{ij}$ for each i and each j.

Exercise 40 *Solve the following matrix equation for a, b, c* and d

$$\begin{pmatrix} a-b & b+c \\ 3d-c & a+d \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ 0 & 3 \end{pmatrix}$$

3.3.4 Matrix arithmetic

In certain circumstances (depending on the size of the matrices) we can add, subtract, and multiply matrices. The rules governing these operations are similar to those for ordinary numbers in many respects, but there are some important differences.

Addition and subtraction of matrices are done element by element, and so the matrices must have the same size. It also means that they must obey the same commutative, associative and distributive laws as ordinary numbers.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same size.

Addition: They can be added to form the sum $A+B=[s_{ij}]$, where $s_{ij}=a_{ij}+b_{ij}$ for each i and each j.

Subtraction: They can be subtracted to form the difference $A-B=[t_{ij}]$ where $t_{ij}=a_{ij}-b_{ij}$ for each i and each j.

Any matrix A can be multiplied by any number λ element by element.

Multiplication by scalars: If λ is any number and A is any matrix, then $\lambda A=[\lambda a_{ij}]$.
In particular, $-1A$ is the *negative* of A, which we denote by $-A$ because $A+(-1A)=0$ the zero vector. We can also write $A-B=A+(-B)$ whenever B is a matrix with the same size as A.

The formula for matrix multiplication is not so natural as it does not simply correspond to multiplication element by element. In fact, in many but not all cases, the matrices that can be multiplied have different sizes. We will not be able to properly motivate the definition of multiplication until we discuss linear transformations, which is done in the next unit. For now we simply note that each term of the product AB is obtained by multiplying a row of A by a column of B element by element and then adding; that is, by taking the dot product of a row of A with a column of B. So each row of A must contain the same number of entries as each column of B. This gives a restriction on the sizes of matrices that can be multiplied.

Matrix Multiplication: If A is an $m\times n$ matrix and B is a $p\times q$ matrix, then the product AB exists if and only if $n=p$, that is, the number of columns of A equals the number of rows of B. If this is true, then AB is an $m\times q$ matrix, and $AB=[c_{ij}]$, where

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj},$$

for each i between 1 and m, and each j between 1 and q. Thus c_{ij} is the dot product of the i^{th} row of A with the j^{th} column of B.

When the product AB exists we say that A and B are **conformable** for multiplication (in that order). If A and B are square matrices then they are conformable in both orders provided that they have the same size. However in general order is important in matrix multiplication. It may be possible to form neither, just one, or both of the products AB and BA. Even if both exist, they may have different sizes. And even if they both exist and have the same size, they may not be equal. To emphasize the importance of the order of the factors, we say that in the product AB, A **pre-multiplies** B, or that B **post-multiplies** A.

Example 17 *Let*

$$A=\begin{pmatrix}1&-2\\-1&2\\-2&4\end{pmatrix},B=\begin{pmatrix}0&-1&1\\0&3&2\\2&1&5\\-2&1&3\end{pmatrix},$$

$$C=\begin{pmatrix}1&1&0\\2&0&1\end{pmatrix},\ and\ D=\begin{pmatrix}10&-2\\0&1\\-2&-3\end{pmatrix}.$$

Of these matrices, the only possible sums and differences are $A+D$, $A-D$, and $D-A$. Thus, for example,

$$A+D=\begin{pmatrix}1+10&-2+(-2)\\-1+0&2+1\\-2+(-2)&4+(-3)\end{pmatrix}=\begin{pmatrix}11&-4\\-1&3\\-4&1\end{pmatrix}.$$

The only possible products are AC, CA, BA, DC, CD and BD. Thus, for example,

$$AC=\begin{pmatrix}1&-2\\-1&2\\-2&4\end{pmatrix}\begin{pmatrix}1&1&0\\2&0&1\end{pmatrix}=\begin{pmatrix}-3&1&-2\\3&-1&2\\6&-2&4\end{pmatrix}$$

and

$$CA = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Exercise 41 *Evaluate the remaining products BA, DC, CD and BD.*

Exercise 42 *Let* $A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 & -1 \\ 6 & 0 & 2 \end{pmatrix}$,
$D = \begin{pmatrix} 1 & 5 & 2 \\ 1 & 4 & -1 \\ 6 & 0 & 2 \end{pmatrix}$, *and* $E = \begin{pmatrix} -1 & 5 & -2 \\ 1 & 0 & -1 \\ 6 & 0 & 2 \end{pmatrix}$
Where possible, find the following:

(i) $D+E$	*(ii)* $(D-E)$	*(iii)* $3D-2E$	*(iv)* AB
(v) DE	*(vi)* ED	*(vii)* $3C-D$	*(viii)* $(3E)D$
(ix) $(AB)C$	*(x)* $A(BC)$	*(xi)* $(4B)C+2B$	*(xii)* $D+E^2$

Example 17 shows that matrices behave somewhat differently to real numbers. Multiplication of real numbers is commutative, that is, the order in which numbers are multiplied is irrelevant. Also if the product of two real numbers is equal to 0, then at least one of the numbers is 0. However, in Example 17 we see that both the products AC and CA exist, but have different sizes, and so cannot be equal. Furthermore, the product CA is a zero matrix, even though neither A nor C is a zero matrix.

Matrix multiplication does however share some of the nice properties of ordinary multiplication. If A, B and C are matrices conformable for multiplication in the order shown, and λ is any number, then

(i) $A(BC) = (AB)C$ (associative law)

(ii) $A(B+C) = AB+AC$ (distributive law)

(iii) $(A+B)C = AC+BC$ (distributive law)

(iv) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

Exercise 43 *Find matrices A and B which show that in general*

$$(A+B)(A-B) \neq A^2 - B^2$$

However, show that if A and B are square matrices with the property that $AB = BA$, *then*

$$(A+B)(A-B) = A^2 - B^2$$

3.3.5 Transpose of a matrix

Interchanging the rows and columns of an $m \times n$ matrix A, gives an $n \times m$ matrix called the **transpose** of A, which we denote by A^T. Formally, if $A = [a_{ij}]$ and $A^T = [c_{ij}]$, then $a_{ij} = c_{ji}$ for each i and j. For square matrices, we can think of A^T as the reflection of A across its main diagonal.

Example 18 *If* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, *then* $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Exercise 44 *For the matrices* A, B, C, D *and* E *as defined in Exercise 42, find*

(i) A^T *(ii)* D^T *(iii)* $AB^T - C^T$ *(iv)* $D^T E^T - (ED)^T$

If $A = A^T$, then we say that A is **symmetric**. Clearly only square matrices can be symmetric. Such matrices occur frequently in applications, especially in the physical sciences.

If A and B are any matrices, conformable for addition or multiplication as required, then

(i) $(A^T)^T = A$,

(ii) $(A+B)^T = A^T + B^T$

(iii) $(AB)^T = B^T A^T$

Notice that the addition of A^T and B^T is defined exactly when the addition of A and B is defined (since A^T and B^T have the same size when A and B have the same size).

Exercise 45 *Explain why $B^T A^T$ is defined when AB is defined.*

Exercise 46 *In exercise 44 we showed that $D^T E^T - (ED)^T = 0$ when the matrices E and D are defined as in Exercise 42. How general is this result? That is, when is $D^T E^T - (ED)^T = 0$ for arbitrary matrices D and E?*

3.4 Solving systems of equations and row reduction

3.4.1 Elimination

The usual way to 'solve' a system of linear equations is to try to 'eliminate' as many of the unknowns by manipulating the equations without changing the solutions. The method is easily demonstrated on the simple system with which we opened this chapter:

$$\begin{array}{rcl} x+y &=& 6 \\ 3x+2y &=& 14 \end{array}.$$

If we subtract 3 times the first equation from the second, then we eliminate the x term in the new second equation:

$$-y = -4.$$

It is now easy to see from the new second equation that $y = 4$ and, if we replace y by 4 in the first equation, that $x = 2$. How can we be sure that this gives the solution to our original pair of equations? Just try it! It certainly gives 'a' solution, and in fact it is the only solution.

What we have done here is transform the original pair of equations to another pair (with the same first equation and a new second equation) which have the same solutions. This has been done so that the solutions of the new equations are easier to find.

It is informative to see what we have done to the augmented matrix $[A : B]$, consisting of the coefficient matrix A of the system together with the forcing vector B as a final column. We start with $[A : B] = \begin{pmatrix} 1 & 1 & 6 \\ 3 & 2 & 14 \end{pmatrix}$ and finish with a new system for which $[A' : B'] = \begin{pmatrix} 1 & 1 & 6 \\ 0 & -1 & -4 \end{pmatrix}$. The new matrix can be obtained from $[A : B]$ by subtracting 3 times its first row from the second - which corresponds exactly to what we did with the equations. Because $[A' : B']$ is a row echelon matrix it is easier to read off the corresponding solutions.

3.4.2 Elementary row operations

It turns out that there are three basic ways that we can manipulate a system of equations without changing its solutions, when they exist. We can:

1. multiply any one of the equations by a non-zero number,
2. change the order of the equations, and
3. add one equation to another.

These operations translate to similar operations on the augmented coefficient matrix of the system. We can:

1. multiply a row by a non-zero constant,

2. interchange two rows, and
3. add one row to another.

These are called the *elementary row operations.*

Row reduction is concerned with using the elementary row operations to transform a matrix to a row echelon or reduced row echelon matrix. Two matrices are said to be **row equivalent** if one can be transformed to the other using only elementary row operations, and it can be shown that any matrix A is row equivalent to a unique reduced row echelon matrix R. We call R the **reduced row echelon form** of A.

Example 19 *Find the reduced row echelon form of the matrix* $A = \begin{pmatrix} 2 & -1 \\ -1 & 4 \\ 1 & 0 \end{pmatrix}$.

Solution We can proceed as follows :

$$\begin{pmatrix} 2 & -1 \\ -1 & 4 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 2 & -1 \end{pmatrix} \begin{matrix} R_3 \\ \\ R_1 \end{matrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 2 & -1 \end{pmatrix} \begin{matrix} \\ R_2+R_1 \\ \\ \end{matrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & -1 \end{pmatrix} \begin{matrix} \\ \\ R_3-2R_1 \end{matrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{matrix} \\ R_2/4 \\ \\ \end{matrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3+R_2 \end{matrix}$$

Notice that the third step actually requires two elementary row operations. The last matrix is the required reduced row echelon form of A.

The **rank** of a matrix is the number of non-zero rows in its reduced row echelon form. As we shall soon see, the number of solutions of a system of equations is determined by the rank of two associated matrices.

Exercise 47 *Find the reduced row echelon forms of the following matrices. Also calculate their ranks.*
(i) $\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$ *(ii)* $\begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix}$ *(iii)* $\begin{pmatrix} -1 & 0 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ *(iv)* $\begin{pmatrix} -1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$
(v) $\begin{pmatrix} 2 & -1 & 1 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$ *(vi)* $\begin{pmatrix} 2 & -1 & 1 & 2 \\ 2 & 2 & 2 & 3 \\ 0 & 3 & 3 & -1 \end{pmatrix}$ *(vii)* $\begin{pmatrix} 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 3 & 3 & -1 \end{pmatrix}$
(viii) $\begin{pmatrix} 2 & -1 & 1 & 2 \\ 2 & 2 & 2 & 3 \\ 0 & 6 & 3 & 3 \end{pmatrix}$ *(ix)* $\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 3 \\ 3 & 3 & -1 \\ 2 & 4 & -1 \end{pmatrix}$

3.4.3 Solving systems of equations

The importance of row reduction to the solution of systems of equations relies on the fact that if the augmented coefficient matrices of two systems are row equivalent, then the systems have precisely the same solutions. Furthermore, it is much easier to find the solutions of a system for which the augmented coefficient matrix is a row echelon or reduced row echelon matrix.

Example 20 *Solve the following system :*

$$\begin{aligned} 2x_1+4x_2-2x_3 &= 2 \\ 4x_1+9x_2-3x_3 &= 8 \\ -2x_1-3x_2+7x_3 &= 10 \end{aligned}$$

Solution Applying elementary row operations to the augmented matrix, we get successively

$$[A:B] = \begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

From the last matrix, which is in row echelon form, we see that the original set of equations is equivalent to the system:

$$\begin{pmatrix} 2x_1 & +4x_2 & -2x_3 & = 2 \\ & x_2 & +x_3 & = 4 \\ & & 4x_3 & = 8 \end{pmatrix}$$

and we can now read off the solutions as $x_3 = 2$ from the last equation, giving $x_2 = 2$ from the second equation, and finally $x_1 = -1$ from the first equation.

Even if the system does not have a unique solution, row reduction is helpful.

Example 21 *Solve the following system :*

$$\begin{aligned} x_1 + x_2 - x_3 &= 8 \\ x_1 - x_2 + 3x_3 &= 2 \end{aligned}$$

Solution Applying elementary row operations to the augmented matrix, we get successively

$$[A:B] = \begin{pmatrix} 1 & 1 & -1 & 8 \\ 1 & -1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & -2 & 4 & -6 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 3 \end{pmatrix}.$$

From the last matrix, which is in row echelon form, we see that the original set of equations is equivalent to a system of 2 equations in the 3 unknowns :

$$\begin{aligned} x_1 + x_3 &= 5 \\ x_2 - 2x_3 &= 3 \end{aligned}$$

This system has many solutions which can be described in many ways. One is to solve for x_1 and x_2 in terms of x_3. Clearly

$$\begin{aligned} x_1 &= 5 - x_3 \\ x_2 &= 3 + 2x_3 \end{aligned}$$

We may regard x_3 as a 'free' variable. Then a typical solution $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ looks like $\mathbf{x} = \begin{pmatrix} 5 & 3 & 0 \end{pmatrix}^T + x_3 \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}^T$, for some real number x_3.

Notice that if we think of x_1, x_2, and x_3 as the components of a vector in $\mathbf{R}^3$, then the solutions $\mathbf{x}$ look like a line in $\mathbf{R}^3$ (compare with equation (2.5) on page 23).

Finally, row reduction makes it obvious when there are no solutions.

Example 22 *Solve the following system :*

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 5x_2 + x_3 &= 10 \end{aligned}$$

Solution Applying elementary row operations to the augmented matrix, we get successively

$$[A:B] = \begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -5 & 1 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & -1 & 12 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

From the last matrix, which is in row echelon form, we see that the original set of equations is equivalent to the system:

$$\begin{pmatrix} 2x_1 & +4x_2 & -2x_3 & = 2 \\ & x_2 & +x_3 & = 4 \\ & & 0 & = 8 \end{pmatrix}$$

However 0 cannot equal 8, and so this system has no solutions.

When the system has no solutions, such as in the last example, we say that it is **inconsistent**.

Exercise 48 *Solve the following systems of equations.*

(i) $\begin{aligned} x+2y &= 6 \\ 2x-y &= 3 \end{aligned}$ *(ii)* $\begin{aligned} 3x-2y &= -6 \\ 2x-5y &= 2 \end{aligned}$ *(iii)* $\begin{aligned} -x+2y &= 1 \\ 2x-y &= 3 \end{aligned}$

(iv) $\begin{aligned} x+5y+z &= 0 \\ 2x+y-2z &= 0 \\ x+7y+2z &= 0 \end{aligned}$ *(v)* $\begin{aligned} x+5y+z &= 2 \\ 2x+y-2z &= 1 \\ x+7y+2z &= 3 \end{aligned}$ *(vi)* $\begin{aligned} x-2y+z &= 2 \\ 2x-y-4z &= 13 \\ x-y-z &= 5 \end{aligned}$

3.5 Row space, rank and the equation $AX = B$

We have seen that a system may have no solutions, exactly one solution, or may have many (in which case we say that the system is **under-determined**). The existence of solutions, as well as the number of solutions, is determined by the **ranks** of two matrices associated with the equations.

We have 'solved[1]' the matrix equation $AX = B$ by performing row reduction on the augmented matrix $[A:B]$ to produce a new system of equations which is simpler to 'solve'. Each row of $[A:B]$ specifies an equation in the new system. The examples in the previous section suggest that if A is an $m \times n$ matrix, so that we are dealing with a system of m equations in n unknowns, then the equation $AX = B$ will have:

- a unique solution precisely when the row echelon form of $[A:B]$ has n non-zero rows with pivot elements in the first n columns.

  In this case, the new system of (non-zero) equations is 'cascading', with the last one involving only x_n, the second last one x_{n-1} and possibly x_n, and so on. So we can 'read off' a unique solution from these equations, obtaining the value for x_n from the last equation, for x_{n-1} from the second last equation and the value already determined for x_n, and so on ....

- no solutions when the row echelon form of $[A:B]$ contains a row of the form $(0,0,0,\ldots,0,\beta)$, where $\beta \neq 0$.

  In the case, the original system of equations is transformed into a new system which is inconsistent because it includes at least one equation of the form $0 = \beta$, where β is non-zero.

[1]We use the word 'solve' rather loosely here as it may include discovering that the system has no solutions.

- many solutions when the row echelon form of $[A:B]$ contains fewer than n non-zero rows, and no row of the form $(0,0,0,\ldots,0,\beta)$, where $\beta \neq 0$.

  In this case, we transform the original system to a smaller consistent system in which we have fewer equations than unknowns.

We can express these results more succinctly in terms of the ranks of A and $[A:B]$. Recall that the rank of a matrix is the number of non-zero rows in its row echelon form. When we perform elementary row operations on a matrix we are simply reordering its rows, or replacing a row by some linear combination of its rows. So the linear span of the rows of a matrix is the same as the linear span of the rows of its row reduced form. We call the linear span of the rows of matrix M the **row space** of M, and denote it by $Row M$. Then what we are observing is that row reduction does not change the row space. Since the dimension of the row space of a matrix in row reduced form is obviously equal to the number of its non-zero rows, it follows that $Row M$ has dimension $rank M$.

It is now easy to see from the second dot point above that the system $AX = B$ has no solutions precisely when the row spaces of the matrices A and $[A:B]$ have different dimensions (because $Row[A:B]$ contains an extra independent row $(0,0,0,\ldots,0,\beta)$, where $\beta \neq 0$. This gives us the following useful result:

Suppose that A is an $m \times n$ matrix.

Theorem 3 (Existence) *The system of equations $AX = B$ has solutions if and only if $rank[A:B] = rank\ A$.*

We call this an *existence theorem* because it explains when a system of equations has solutions.

Suppose now that we know that the system $AX = B$ has a solution (so that $rank\ [A:B] = rank A$). From the first dot point, we see that it will have exactly one solution when the row echelon form of $[A:B]$ has n non-zero rows, and each of these rows has at least one non-zero entry in its first n positions. This means that $Row A$ must have dimension n, which leads to our second important result, a *uniqueness theorem*.

Theorem 4 (Uniqueness) *If the system of equations $AX = B$ has solutions, then the solution is unique if and only if $rankA = n$, the number of columns of A.*

Example 23 *Consider the system of equations*

$$\begin{aligned} x+y &= 6 \\ 2x+2y &= a \end{aligned}$$

where a is a constant. For which a does the system have a solution?

Solution Here $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, and $[A:B] = \begin{pmatrix} 1 & 1 & 6 \\ 2 & 2 & a \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 1 & 6 \\ 0 & 0 & a-12 \end{pmatrix}$. Clearly rank$A = 1$, but rank$[A:B] = 2$ if $a \neq 12$. So the system is inconsistent if $a \neq 12$. On the other hand, if $a = 12$, then rank$[A:B] = 1$, (the second equation is a multiple of the first), and the system has many solutions - we have $y = 6 - x$ and x is arbitrary.

Exercise 49 *Determine for what values of k each system has (a) a unique solution, (b) no solution and (c) many solutions.*

(i) $\begin{aligned} 3x+2y &= 1 \\ 6x+4y &= k \end{aligned}$ *(ii)* $\begin{aligned} 3x+2y &= 0 \\ 6x+ky &= 0 \end{aligned}$ *(iii)* $\begin{aligned} -x+2y &= 1 \\ 2x-y &= k \end{aligned}$

(iv) $\begin{aligned} x+5y+z &= 0 \\ 2x+y-2z &= k \\ x+7y+2z &= 0 \end{aligned}$ *(v)* $\begin{aligned} x-2y+z &= 2 \\ 2x-y-4z &= 13 \\ x-ky-z &= 5 \end{aligned}$

3.5.1 Using rank to check linear independence

Our task here is to decide whether m vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ in $\mathbf{R}^n$ are linearly independent. For example, are the vectors $\mathbf{a}_1 = (1,-2,3,4), \mathbf{a}_2 = (2,-3,1,0)$ and $\mathbf{a}_3 = (1,1,1,1)$ linearly independent in $\mathbf{R}^4$? This will be the case if the linear span of the vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ is 3 dimensional. Since row reduction does not change the row space of a matrix, we can easily find the dimension of this linear span by applying row reduction to the matrix A whose rows are $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$; that is, to

$$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to check that a row echelon form of A is $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -5 & -8 \\ 0 & 0 & -17 & -28 \end{pmatrix}$, and so the rank of A is 3, and its rows are linearly independent.

More generally we see that:

Theorem 5 *The rows of an $m \times n$ matrix A are linearly independent if and only if $rank A = m$, the number of rows of A.*

Example 24 *Decide whether or not the vectors $(1,-2,3)$, $(2,-3,4)$ and $(1,-3,5)$ are linearly independent in $\mathbf{R}^3$.*

Solution Writing the vectors as rows of a matrix gives $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -3 & 4 \\ 1 & -3 & 5 \end{pmatrix}$ which is row equivalent to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$. So $rank\,A = 2$, and the vectors are not linearly independent.

Note that since the row space of matrices is unchanged by row reduction, we can easily produce a basis for the linear span of the vectors $(1,-2,3), (2,-3,4)$ and $(1,-3,5)$ given in the last example by picking out the non-zero rows of the row echelon form of A. So $\{(1,0,-1),(0,1,-2)\}$ is a basis. More generally, the non-zero rows of the row echelon form of the matrix with rows $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_m$ form a basis for the linear span of these vectors.

Exercise 50 *Use matrices to test the following sets of vectors for linear independence*

(i) $(1,2,5)$, $(2,3,7)$ *and* $(0,4,7)$.

(ii) $(1,2,5)$, $(-2,3,7)$ *and* $(1,-5,-12)$.

(iii) $(1,2,5,3)$, $(-2,3,7,6)$ *and* $(1,-5,-12,5)$.

For those that are linearly dependent, find a basis for their linear span.

We observed in section 2.6.2 that a collection of 4 or more vectors in $\mathbf{R}^3$ must be linearly dependent. In fact any collection of $n+1$ vectors in $\mathbf{R}^n$ must be linearly dependent. Theorem 5 provides a reason. If $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$ are vectors in $\mathbf{R}^n$, then the matrix A whose rows are $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$ cannot have rank greater than n. So rank$A = m$ is possible only if $m \leq n$.

3.6 Matrix inversion

As suggested at the start of this chapter, for a system in which there are the same number of equations as unknowns, so that the coefficient matrix is square, it may be useful to find the inverse of the coefficient matrix. This is just one useful application of a matrix inverse.

3.6.1 Inverse of a matrix

In this section we only consider square matrices, and we look for the matrix equivalent of a reciprocal. This is an important step towards applying matrix theory to the solution of systems of linear equations.

Given a square matrix A, a matrix B which satisfies $AB = I = BA$ is called the **inverse** of A, and is denoted by A^{-1}. While every non-zero number has a reciprocal (inverse), the situation is not so simple for matrices. For example, you should already know that a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse if and only if the number $\Delta = ad - bc$ is non-zero. The number Δ is called the **determinant** of A .

Exercise 51 *Show by multiplying out that*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \Delta^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ when } \Delta \neq 0,$$

and hence write down a non-zero 2×2 *matrix which has no inverse.*

If it exists, B will have the same size as A. It turns out that the conditions $AB = I$ and $BA = I$ are equivalent when A and B are square matrices: if one is true, the other must be. So in order to check that B is the inverse of A, it is enough to check either one of the products AB or BA.

A matrix that has an inverse is called **invertible**, while one that does not have an inverse is called **non invertible**. A matrix that is not invertible is sometimes called **singular**, while an invertible matrix is sometimes called **non-singular**.

It is easy to show that if A and B are $n \times n$ matrices with inverses, then

$$(A^{-1})^{-1} = A \text{ and } (AB)^{-1} = B^{-1}A^{-1}$$

Notice the change of the order of the factors here: in $(AB)^{-1}$, B^{-1} pre-multiplies A^{-1}.

In many of the applications of matrix theory, it is necessary to decide firstly whether a particular matrix has an inverse, and then to find the inverse if it exists. We have seen how this can be done for 2×2 matrices, provided we can remember the formula. For larger matrices the task initially seems quite daunting. However, row reduction helps us here. It will produce the inverse whenever it exists, and you don't have to remember any formulae. This is particularly important because, although many computer packages will compute inverses, they do not always get the 'correct' answer. For example, a matrix may be invertible but because of rounding error the computer claims that it is not. The row reduction process can shed light on why a matrix does not have an inverse, why the computer software thinks it does not have an inverse when it does, or why the computer returns an answer that is not a good approximation to the inverse.

3.6.2 Finding inverses using row reduction

Suppose that A is an $n \times n$ matrix, and let $[A : I]$ denote the $n \times 2n$ matrix obtained by writing A and the $n \times n$ identity matrix I side by side, with A first. Thus, if $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ then $[A : I] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}$. It can be shown that

if the row echelon form of $[A : I]$**is** $[I : B]$, **then** $B = A^{-1}$.

This is the key to finding inverses using row reduction.

Example 25 *Find the inverse of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.

Solution We have $[A : I] = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{pmatrix}$. Proceeding with our row reduction, we get the successive row equivalent matrices

$$[A : I] \to \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix}.$$

So $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$. This can be checked by multiplying with A.

Exercise 52 *Find the inverses of the following matrices using row reduction.*
(i) $\begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}$ *(ii)* $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ *(iii)* $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ *(iv)* $\begin{pmatrix} 4 & 3 & 1 \\ 0 & 2 & 6 \\ 1 & 0 & 1 \end{pmatrix}$

Exercise 53 *Show that the inverse of* $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}$ *is* $\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -c+ab & -b & 1 \end{pmatrix}$. *Does this inverse exist for all values of a, b, and c?*

It is natural to ask what happens if the reduced row echelon form of $[A:I]$ is not of the form $[I:B]$. This occurs precisely when A has no inverse! So the reduced row echelon form of $[A:I]$ gives us the inverse A^{-1} if it is there to be found, and tells us when there is no inverse. We also see that if A has an inverse then its reduced row echelon form must be I, the identity matrix. Conversely, if the reduced row echelon form of A is I then A is invertible, and so A is invertible if and only if $rank\, A = n$, the size of A.

Exercise 54 *By using row reduction and/or a computer package, find the inverse of the matrix* $\begin{pmatrix} a & 2 & 3 \\ 0 & 3 & 5 \\ 1 & 5 & 8 \end{pmatrix}$ *when it exists.*

Exercise 55 *The matrix X given below was placed into a computer software package called Scientific Notebook and the inverse was given to be the matrix Y shown below. Calculate XY (either using a computer package or by hand) and comment on the result. Find the inverse of X by hand and compare it with Y. By examining the steps performed in your row reduction and by using the fact that the computer stores numbers as a decimal approximation, explain why there is a difference between the exact inverse and the inverse given by the computer.*

$$X = \begin{pmatrix} 1 & 2 \\ 0.0005 & 0.00999 \end{pmatrix} \qquad Y = \begin{pmatrix} 1.1112 & -222.47 \\ -0.055617 & 111.23 \end{pmatrix}$$

3.6.3 Using inverses to solve systems of equations

We again return to the system

$$\begin{aligned} x+y &= 6 \\ 3x+2y &= 14 \end{aligned}$$

or equivalently $AX = B$,where $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$. For the coefficient matrix A in this example, we find that $\Delta = -1$, and so A is invertible and $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$. So the system can be solved by multiplying on the left of both sides of the matrix equation by A^{-1} :

$$X = (A^{-1}A)X = A^{-1}(AX) = A^{-1}B.$$

Thus

$$X = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 14 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

that is, $x = 2$ and $y = 4$.

Exercise 56 *For each system in Exercise 48, decide whether the coefficient matrix is invertible and solve the system using the inverse where possible.*

3.7 Determinants

The determinant of a matrix is a single number that contains a large amount of information. For example, it tells immediately whether a matrix is invertible!!

You already know that a 1×1 matrix $[a]$ (that is, a number a) is singular (non-invertible) if and only if $a=0$. It is not so trivial, but still easy, to show that the 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is singular if and only if the **determinant** $\Delta = ad-bc$ is zero. So in these cases there is an associated number which vanishes if and only if the matrix is singular. The amazing thing is that there is such a number for all square matrices, and we also call that number the determinant of the matrix. Its definition is rather intricate, and it turns out to be easiest to define the determinant of a 3×3 matrix in terms of the determinants of certain 2×2 matrices, then to define the determinant of a 4×4 matrix in terms of the determinants of certain 3×3 matrices, and so on.

3.7.1 Determinants of $n\times n$ matrices

The basic fact is that no matter what the size of the matrix, its determinant is a sum of signed products (that is, each preceded by either a $+$ or a $-$ sign), where each product contains exactly one term from each row and column of the matrix.

The formula for the determinant of a 3×3 matrix $A=[a_{ij}]$ is not too difficult to write down. It is made up of six such products:

$$\det A = a_{11}(a_{22}a_{33}-a_{32}a_{23}) - a_{12}(a_{21}a_{33}-a_{31}a_{23}) + a_{13}(a_{21}a_{32}-a_{31}a_{22}). \tag{3.1}$$

We can see that each product contains exactly one term from each row and column of the matrix because each number $1,2$ and 3 appears just once in each of the subscript positions. We can in fact determine the sign of associated with each product by examining the order in which the numbers appear in the subscripts, but we will not pursue that here.

The formula (3.1) for the determinant can be expressed in terms of determinants of smaller matrices (*submatrices* of A):

$$\det A = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Here we have picked out the elements of the first row of A and multiplied each by the determinant of a 2×2 matrix which is actually the submatrix of A that we get by deleting the first row and the j^{th} column which contains the corresponding element a_{1j}. Note that the sign of the middle term is a minus, but we'll say more about this later.

It will be useful to adopt the convention of replacing the round brackets about a matrix by vertical bars when we want to indicate its determinant. Thus

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc.$$

Example 26 *Find the determinant of the matrix* $A=\begin{pmatrix} 1 & -2 & 4 \\ 3 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

Solution Picking out the elements of the first row, we have

$$\begin{vmatrix} 1 & -2 & 4 \\ 3 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} \begin{aligned} &= (1)\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} + (4)\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} \\ &= (1)(-2)-(-2)(-3)+(4)(3) \\ &= 4 \end{aligned}$$

The determinant of an n by n matrix A is defined to be

$$\det A = a_{11}M_{11} - a_{12}M_{12} + \ldots + (-1)^{n+1}a_{1n}M_{1n} \tag{3.2}$$

where a_{1j} is the j^{th} entry of the first row of A and M_{1j} is the determinant of the $n-1$ by $n-1$ matrix formed by removing the first row and j^{th} column from A (the M_{1j} are called **minors** of A). Note that the signs of the terms alternate between 1 and -1.

In equation (3.2), we have defined the determinant in terms of the entries of the first row of A. We say that we are calculating $\det A$ by *expanding across the first row* of A. However, this is not the only way to do it. There are similar (equivalent) definitions of determinant which expand across any row or indeed down any column of A. The associated minors are still determined by deleting the row and column containing the 'coefficient'. The slight complication is that the signs of the terms vary according to the following 'sign rule': the sign preceding a particular matrix entry in the definition of the determinant is obtained from the corresponding entry of the matrix:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{pmatrix}$$

When evaluating a determinant by hand, a good strategy is to expand across the row or column with the greatest number of zero entries, because if a term a_{ij} is zero then we do not need to calculate the corresponding minor.

Even so, for matrices larger than 3 by 3 it is nearly always better to use a computer to calculate determinants, because for example to find the determinant of a 5 by 5 matrix we must find the determinant of up to 5 matrices of size 4 by 4 (and to find each of these we must find the determinant of up to 4 matrices of size 3 by 3!). Important exceptions include recognizing at a glance matrices that have a determinant of zero (such as when a row consists entirely of zeros, or when two rows or columns are equal) and therefore are not invertible (see section 3.7.4), or when we wish to explore the structure of a class of matrices (such as showing that the determinant of a lower triangular matrix of any size is always the product of the diagonal entries).

Exercise 57 *Find* $\begin{vmatrix} 7 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 & 0 \\ 6 & 0 & 4 & 2 & 0 \\ 4 & a & -6 & 3 & c \\ 3 & b & 4 & 7 & d \end{vmatrix}$.

3.7.2 Properties of determinants

We now list a number of properties of determinants. These can all be easily verified for 2×2 matrices, and in many cases it is easy to see why they also hold for larger matrices when we recall the basic structure of the determinant as a sum of signed products.

Property 1 : The determinant of a triangular matrix is the product of its diagonal entries, and in particular $\det I = 1$.

This is obvious since the product of the diagonal entries is the only allowed product that can be non-zero for a triangular matrix; all others contain at least one zero factor.

Property 2 : If the matrix B is obtained by multiplying a single row of A by a constant λ, then $\det B = \lambda \det A$.

Again obvious because multiplying a single row of A by λ multiplies each product in the determinant by λ.

Property 3 : If A has a row consisting entirely of zeros, then $\det A = 0$.

Each product in the determinant must contain a zero factor since it must contain a term from the given row.

Property 4 : If B is obtained by interchanging two rows of A, then $\det B = -\det A$.

Interchanging two rows does not change the products that appear in the determinant, just the signs of those products. This property is a consequence of that change.

Property 5 : If B is obtained by adding a multiple of one row of A to another row of A, then $\det A = \det B$.

Property 6 : If two rows of A are equal, then $\det A = 0$.

If two rows are equal then we can add one of the rows to the negative of the other without changing the determinant by Property 5, but we get a matrix which has a zero row, and so zero determinant by Property 3.

Property 7 : If matrices A, B and C are identical except for a single row, and if the exceptional row of C is the sum of the corresponding rows of A and B, then $\det C = \det A + \det B$.

Property 8 : For any (square) matrix A, $\det A^T = \det A$.

This is because our definition of determinant really does not distinguish between rows and columns, and the rows and columns of A and A^T are simply interchanged.

Property 9 : For any two square matrices A and B of the same size,

$$\det AB = (\det A)(\det B).$$

This is a very important property that we can't prove here. It is proved for 2×2 matrices in Example 3.7.2.

Furthermore, properties 2 to 7 are also true if we consider columns instead of rows because of property 8 and the fact that the rows of A^T equal the columns of A. For example if A has a column consisting entirely of zeros then A^T has a row consisting entirely of zeros so by properties 8 and 3 $\det A = \det A^T = 0$.

As we have seen, many of the properties follow easily from the definition of the determinant. Those that don't can be easily checked for 2×2 matrices.

Example 27 *Verify Property 5 for 2 by 2 matrices.*

Solution: Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $B = \begin{pmatrix} a+\lambda c & b+\lambda d \\ c & d \end{pmatrix}$ for some number λ. Then

$$\det B = (a+\lambda c)d - (b+\lambda d)c = ad - bc = \det A.$$

A similar argument works if we add a multiple of the first row to the second.

Example 28 *Verify Property 9 for 2 by 2 matrices.*

Solution: Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then $AB = \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$ and so

$$\begin{aligned} \det AB &= (aw+by)(cx+dz) - (ax+bz)(cw+dy) \\ &= awdz + bycx - axdy - bzcw \\ &= (ad-bc)(wz-xy) \\ &= \det A \det B. \end{aligned}$$

Exercise 58 *Verify Property 7 for 2 by 2 matrices.*

Exercise 59 *Explain how it follows from Property 2 that for a 2×2 matrix, $\det \lambda A = \lambda^2 \det A$. What is the corresponding formula for $n\times n$ matrices.*

You should not confuse Property 7 with the statement $\det(A+B) = \det A + \det B$ which is not usually true.

Exercise 60 *Find 2×2 matrices A and B which show that $\det(A+B)$ need not equal $\det A + \det B$.*

Exercise 61 *Verify that, for any 2×2 matrix A, $\det A^T = \det A$*

Exercise 62 *Suppose that $A^2 = A$ where A is a square matrix. Use property 9 of determinants to show that either $|A| = 0$ or $|A| = 1$.*

Exercise 63 *Use Property 9 to show that $\det A^{-1} = (detA)^{-1}$*

3.7.3 Elementary row operations and determinants

Property 2 states that the elementary row operation of multiplying a row by a scalar, multiplies the determinant by the same factor, and Property 4 states that the elementary row operation of interchanging rows simply changes the sign of the determinant. Finally, Property 5 states that the third elementary operation of adding a multiple of one row to another leaves the determinant unchanged.

These observations, together with the ease with which we can calculate determinants of triangular matrices (Property 1), enable us to use row reduction to calculate the determinant of a matrix. We must simply keep track of the row operations that we perform.

Example 29 *Find the determinant of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.

Solution Proceeding with our row reduction, we get the successive row equivalent matrices

$$A \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 3 & 5 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{pmatrix},$$

where we first subtracted twice the first row from the second and three times the first from the third, and then simply interchanged the second and third rows. Thus we see that the reduced row echelon form of A has determinant $1 \times (-1) \times (-1) = 1$ and that $\det A$ is equal to -1 times this; that is $\det A = -1$.

3.7.4 Determinants and invertibility

We started our discussion of determinants with the observation that determinants were important because they provide a single number which indicates whether or not, at least in the 2×2 case, a matrix A is invertible. The general result is as follows

Theorem 6 *A square matrix A is invertible if and only if* $\det A \neq 0$.

Proof Suppose that A is an $n \times n$ matrix. We know that A is invertible if and only if it has rank n; that is, if and only if any row echelon matrix to which it is row equivalent has n non-zero rows. However, an $n \times n$ row echelon matrix with n non-zero rows must have n non-zero diagonal entries (can you see why?), and so has a non-zero determinant - remember that Property 1 guarantees that the determinant of a square row echelon matrix is the product of its diagonal elements. Since the elementary row operations can only change the sign of a determinant or multiply it by a non-zero factor, this means that A is invertible if and only if $\det A \neq 0$.

Exercise 64 *Decide whether each of the following matrices is invertible by calculating the determinant, and find the inverses of those that are.*

(i) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ *(ii)* $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ *(iii)* $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & -1 & 4 \\ -1 & 0 & 5 \\ 19 & -7 & 3 \end{pmatrix}$ *(v)* $\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$

Exercise 65 *Use determinants to decide whether the given homogeneous system has a unique solution or an infinite number of solutions.*

(i) $\begin{aligned} 2x + y &= 0 \\ -x + 6y &= 0 \\ 4x + 2y + z &= 0 \end{aligned}$ *(ii)* $\begin{aligned} x + y + z &= 0 \\ -2x - 3y + 4z &= 0 \\ x + 7z &= 0 \end{aligned}$

3.8 $AX = B$ for the last time ...

We have seen that matrices can be used to solve systems of simultaneous equations of the form $A\mathbf{X} = B$, where A is an n by m matrix, $\mathbf{X}$ is a column vector containing the m unknowns $x_1, \ldots, x_m$ and $\mathbf{B}$ is a fixed column vector containing $b_1, \ldots, b_n$. We have also seen that there may be no solutions, a unique solution, or there may be many solutions to the system, and we have seen how the ranks of the coefficient matrix A and the augmented matrix $[A : B]$ may be used to decide which is true for a given system of equations.

In this section we see how to efficiently describe all solutions of a given system. Of course, if the system has no solutions then there is nothing to do. In the other cases, the form of the solution that we give is called the *general solution* to the equation because it contains all possible solutions.

We first consider the case where $B = \mathbf{0}$, the zero vector. This gives the *homogeneous* equation $AX = \mathbf{0}$. Its solutions are called *homogeneous solutions*, and they form the **null space** of the matrix A. So the elements of the null space of A are precisely those column vectors X in $\mathbf{R}^m$ which give $\mathbf{0}$ when multiplied by A on the left. The null space is always non-empty because the zero vector is in it, and is a subspace of $\mathbf{R}^m$ because the sum of any two of its elements is another element, as is any scalar multiple of an element. Therefore we can use a basis for the null space to describe it efficiently - every element of the null space is a linear combination of the basis elements.

Once we have described the homogeneous solutions it is easy to describe the solutions to $AX = B$ for any forcing vector B. To see why, suppose that X_1 is a solution of the general equation (that is, that $AX_1 = B$), and that X_0 is any vector in the null space of A. Then

$$A(X_0 + X_1) = AX_0 + AX_1 = \mathbf{0} + B = B,$$

and so $X_0 + X_1$ is also a solution to $AX = B$. In fact *every* solution of $AX = B$ can be expressed as a sum $X_0 + X_1$, where X_0 is a vector in the null space of A and X_1 is the one *particular solution* of $AX = B$ that we have found (it can be any solution at all, but the same one can be used for every solution!!). Indeed, if $AX_1 = B = AX_2$ then

$$A(X_2 - X_1) = AX_2 - AX_1 = 0$$

shows that $X_2 - X_1 = X_0$ belongs to the null space of A. In other words, we can describe *all* solutions to $AX = B$ using a particular solution and the null space of A[2]. This leads to the *general solution* of the system, which includes all possible solutions.

Example 30 *If we use row reduction to find the solution of*

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

we see that

$$X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - 3y \\ y \\ \frac{1}{2} - 2t \\ t \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

for any y and t. This is a general form of the solution. It is easy to check that $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}^T$ is a particular solution, satisfying $AX = B$, and that $\begin{pmatrix} -3 & 1 & 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & -2 & 1 \end{pmatrix}^T$ form a basis for the null space of A.

Exercise 66 *In example 21 on page 40 we discovered that the solutions to $x_1 + x_2 - x_3 = 8$ and $x_1 - x_2 + 3x_3 = 2$ were $\mathbf{x} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ for some real number t.*

What are the solutions to $x_1 + x_2 - x_3 = 0$ and $x_1 - x_2 + 3x_3 = 0$?

Exercise 67 *Find the general solutions to $x + 3y + 3z = 1$, $2x + 6y + 9z = 5$ and $-x - 3y + 3z = 5$?*

[2] This approach to describing the solutions is also employed in other applications (such as in the solution to differential equations).

Solutions for Chapter 3

Exercise 39

A is upper triangular and a row echelon matrix. B is square, upper triangular, and a row echelon matrix. C and D are upper triangular, row echelon, and reduced row echelon matrices.

Exercise 40

$a = 2, b = 5, c = 3$, and $d = 1$

Exercise 41

$$BA = \begin{pmatrix} -1 & 2 \\ -7 & 14 \\ -9 & 18 \\ -9 & 18 \end{pmatrix}$$

$$DC = \begin{pmatrix} 6 & 10 & -2 \\ 2 & 0 & 1 \\ -8 & -2 & -3 \end{pmatrix}$$

$$CD = \begin{pmatrix} 10 & -1 \\ 18 & -7 \end{pmatrix}$$

$$BD = \begin{pmatrix} -2 & -4 \\ -4 & -3 \\ 10 & -18 \\ -26 & -4 \end{pmatrix}$$

Exercise 42

(i) $\begin{pmatrix} 0 & 10 & 0 \\ 2 & 4 & -2 \\ 12 & 0 & 4 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 5 & 5 & 10 \\ 1 & 12 & -1 \\ 6 & 0 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 9 & 11 \\ 6 & 9 \\ 3 & 4 \end{pmatrix}$

(v) $\begin{pmatrix} 16 & 5 & -3 \\ -3 & 5 & -8 \\ 6 & 30 & -8 \end{pmatrix}$

(vi) $\begin{pmatrix} -8 & 15 & -11 \\ -5 & 5 & 0 \\ 18 & 30 & 16 \end{pmatrix}$

(vii) not defined

(viii) $\begin{pmatrix} -24 & 45 & -33 \\ -15 & 15 & 0 \\ 54 & 90 & 48 \end{pmatrix}$

(ix) $\begin{pmatrix} 75 & 36 & 13 \\ 60 & 24 & 12 \\ 27 & 12 & 5 \end{pmatrix}$
(x) $\begin{pmatrix} 75 & 36 & 13 \\ 60 & 24 & 12 \\ 27 & 12 & 5 \end{pmatrix}$
(xi) not defined
(xii) $\begin{pmatrix} -5 & 0 & -5 \\ -6 & 9 & -5 \\ 12 & 30 & -6 \end{pmatrix}$

Exercise 43

$(A+B)(A-B) = A^2 + BA - AB - B^2$ so $(A+B)(A-B) = A^2 - B^2$ if $AB = BA$. Any matrices such that $AB \neq BA$ will not satisfy the equality, such as $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Exercise 44

(i) $\begin{pmatrix} 3 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$(ii) $\begin{pmatrix} 1 & 1 & 6 \\ 5 & 4 & 0 \\ 2 & -1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 12 & -7 \\ -2 & 1 \\ 4 & -2 \end{pmatrix}$ (iv) $\mathbf{0}$, i.e. the 3×3 zero matrix.

Exercise 45

AB is defined when the number of rows of B equals the number of columns of A, which is when the number of columns of B^T equals the number of rows of A^T, which is when $B^T A^T$ is defined.

Exercise 46

If the number of rows of D equals the number of columns of E, then the matrix multiplications and addition in $D^T E^T - (ED)^T$ are defined and this expression is the zero matrix (since $(AB)^T = B^T A^T$ when AB exists).

Exercise 47

(i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 7/2 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 7/4 \end{pmatrix}$
(v) $\begin{pmatrix} 1 & 0 & 2/3 & 7/6 \\ 0 & 1 & 1/3 & 1/3 \end{pmatrix}$ (vi) $\begin{pmatrix} 1 & 0 & 0 & 11/6 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ (vii) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
(viii) $\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ (ix) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.
(i) to (v) have rank2 while (vi) to (ix) have rank 3.

Exercise 48

(i) $x = 12/5, y = 9/5$ (ii) $x = -34/11, y = -18/11$
(iii) $x = 7/3, y = 5/3$ (iv) $x = 0, y = 0, z = 0$
(v) $x = 4, y = -1, z = 3$ (vi) $x = 3z + 8, y = 2z + 3, z$ arbitrary

Exercise 49

(i) no solution if $k \neq 2$,many solutions if $k = 2$ (ii) unique solution if $k \neq 4$, many solutions if $k = 4$ (iii) unique solution for all k (iv) unique solution for all k (v) unique solution if $k \neq 1$, many solutions if $k = 1$

Exercise 50

(i) independent (rank=3) (ii) dependent (rank=2) (iii) independent (rank=3)

Exercise 51

This result follows from the fact that the product of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is a matrix with Δ on the diagonals and 0 otherwise. Any matrix where $ad = bc$ has no inverse, for example the 2 by 2 matrix with all entries equal to 0.

Exercise 52

(i) $\begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} -1/2 & 3/2 \\ 1 & -2 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
(iv) $\begin{pmatrix} 1/12 & -1/8 & 2/3 \\ 1/4 & 1/8 & -1 \\ -1/12 & 1/8 & 1/3 \end{pmatrix}$

Exercise 53

The inverse exists for all a, b, and c (this can be shown directly by showing that the matrix multiplication of the matrix by its inverse is the identity matrix, or by showing that the determinant is never 0).

Exercise 54

The inverse is $\begin{pmatrix} 1 & 1 & -1 \\ -5 & 3-8a & 5a \\ 3 & 5a-2 & -3a \end{pmatrix} / (a-1)$ when $a \neq 1$ and has no inverse when $a = 1$. Many people are often sloppy here and imply that the inverse always exists. Despite the fact that computer packages usually expect you to be very precise about what you mean, many are sloppy here too.

Exercise 55

The product XY is $\begin{pmatrix} 0.99997 & -0.01 \\ -1.383 \times 10^{-8} & 0.99995 \end{pmatrix}$. This product should equal the identity, however one entry is -0.01 which is not very close to 0. The exact inverse is $\begin{pmatrix} 0.00999 & -2 \\ -0.0005 & 1 \end{pmatrix} / 0.00899$. After performing the division here and rounding the results we obtain the result given by the computer. Unfortunately, for the second column of the inverse this rounding is only accurate to about 0.01. This appears to be accurate at first because it represents a reasonable number of significant figures but it is the number of accurate decimal places that is important here. Do not forget that most computer packages have ”bugs” (or give the ”wrong” answers). Sometimes there is a logical reason for this, such as the rounding here, but sometimes the package has been written to do the ”wrong” thing (or at least something different to what you expected).

Exercise 57

Repeated use of equation (3.2) gives the determinant as $7 \begin{vmatrix} 0 & 8 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ a & -6 & 3 & c \\ b & 4 & 7 & d \end{vmatrix} = 7(-8) \begin{vmatrix} 0 & 2 & 0 \\ a & 3 & c \\ b & 7 & d \end{vmatrix} =$
$7(-8)(-2) \begin{vmatrix} a & c \\ b & d \end{vmatrix} = 112(ad - bc)$.

Exercise 59

For a 2 by 2 matrix A, λA is the matrix formed by multiplying the first row of A by λ and then multiplying the second row by λ. Applying property 2 twice gives $\det \lambda A = \lambda^2 \det A$.

The corresponding result for an n by n matrix is $\det \lambda A = \lambda^n \det A$.

Exercise 60

This is true for most matrices. Take for example A and B both equal to the identity matrix. $\det A$ and $\det B$ equal 1 but $\det(A+B) = 4$.

Exercise 61

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{and} \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

both equal $ad - bc$.

Exercise 62

$|A| = \left|A^2\right| = |A|\,|A| = |A|^2$. Since the only real numbers that are equal to their square are 0 and 1 it follows that $|A| = 0$ or $|A| = 1$.

Exercise 64

(i) determinant $= 4$, inverse $= \begin{pmatrix} 1/2 & -1/2 \\ -1/4 & 3/4 \end{pmatrix}$

(ii) -12, $\begin{pmatrix} 1/3 & -1/4 & -1/6 \\ 0 & 1/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{pmatrix}$

(iii) -1 $\begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

(iv) 0, no inverse

(v) -2, $\begin{pmatrix} 1 & -2 & -5 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & -1 \end{pmatrix}$

Exercise 65

(i) The determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 6 & 0 \\ 4 & 2 & 1 \end{pmatrix}$ is 13 (note that the determinant of A^T is just the determinant of a 2 by 2 matrix so is easy to do). Therefore there is a unique solution to these equations (with all three variables equal to zero).

(ii) The determinant of $A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -3 & 4 \\ 1 & 0 & 7 \end{pmatrix}$ is 0 so there are an infinite number of solutions. (If you got a determinant of -35 then check the last row in your matrix!).

Exercise 66

$\mathbf{x} = t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ for some real number t. This is the null space of the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.