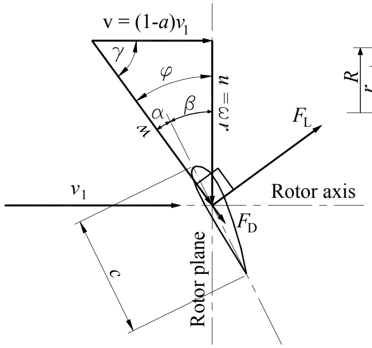
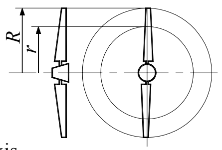
Wind Turbines

- Design of an optimal rotor i.e. pitch angle and chord length of the blades and how to calculate the power production
- Including a spread sheet model in Excel and an EES-model for the calculations





2. Edition

Søren Gundtoft
University College of Aarhus
June 2009

Foreword

This paper (now 2. edition) deals with the general procedures for the design of the rotor blades – pitch angle and chord length – for horizontal axis wind turbines (HAWT) and for the calculation of power production in the rotor.

The paper is used in the course Fluid Dynamics (code: MIFLD1) at the Department of Mechanical Engineering at the University College of Århus.

If you find errors in this report, you should send them to me, mail to: sgt@iha.dk.

In this paper a comma is used as decimal point, i.e. $\pi = 3{,}1416$.

Compared to the 1. Edition (2004) Chapter 3 and Chapter 6 have been totally rewritten.
Also an EES-model of the BEM method is included – see appendix C

June 2009 / Søren Gundtoft

Content

1. Introduction

It is assumed that the reader knows some basic fluid mechanics, for example the basic theories of fluid properties, the ideal gas law, Bernoulli's equation, turbulent and laminar flow, Reynolds's number, etc. Important for the understanding of the theory is mastering the conservation laws for momentum and angular momentum, for which reason these theories are presented in Appendix A.

In this paper you will find:

- Chapter 2: How much energy that can be taken from the wind in an idealized wind turbine (proof of Betz' law) and how to design the rotor
- Chapter 3: How to design a an optimal rotor – pitch angle and chord length
- Chapter 4: Characteristic of rotor blades (coefficients of lift and drag)
- Chapter 5: How to calculate the power of a given rotor (the BEM theory)
- Chapter 6: Efficiency of a wind turbine
- Chapter 7: An example – BEM method in Excel
- Chapter 8: Distribution of the natural wind and calculation of annual energy production

Most of the theory is taken from ref./1/ and /4/. Data for rotor blade sections are taken from ref./2/ and /3/.

All important calculations are demonstrated by examples. The calculations after the BEM method can be done by simple spread sheet calculations and in Appendix B the formulas are printed. Also the code for an EES-model of the BEM method is presented, see Appendix C.

2. Power in the wind

The question is: How much energy can be taken from the wind? The wind turbine decelerates the wind, thereby reducing the kinetic energy in the wind. But the wind speed cannot be reduced to zero – as a consequence, where should the air be stored? As first time shown by Betz, there is an optimum for the reduction of the wind speed, and this is what is to be outlined in this chapter. Figure 2.1 shows the streamlines of air through a wind turbine

Notation: In the following we will use index 1 for states "far up stream" the rotor plane, index for the states in the rotor plane and index 3 for states far downstream. For simplicity index 2 – states in the rotor plane - will be omitted in most cases.

Long in front of the rotor, the wind speed is v_1. After passing the wind turbine rotor (called the rotor in the following), the wind speed would be reduced to v_3. The pressure distribution is as follows. The initial pressure is p_1. As the air moves towards the rotor, the pressure rises to a pressure p_+ and by passing the rotor, the pressure suddenly falls by an amount of Δp i.e. the pressure is here $p_- = p_+ - \Delta p$. After passing the rotor, and far down stream the pressure again rises to $p_3 = p_1$. Curves for wind speed and pressure are shown in figure 2.1.

Bernoulli's equation: If we look at the air moving towards the rotor plane, we can use the Bernoulli's equation to find the relation between the pressure p and the speed v, while we can make the assumption that the flow is frictionless:

$$\frac{1}{2}\rho v^2 + p = p_{tot} \quad \text{[Pa]} \quad (2.1)$$

where p_{tot} is the total pressure, which is constant. That means, if the speed of flow goes up, the pressure goes down and vice versa.

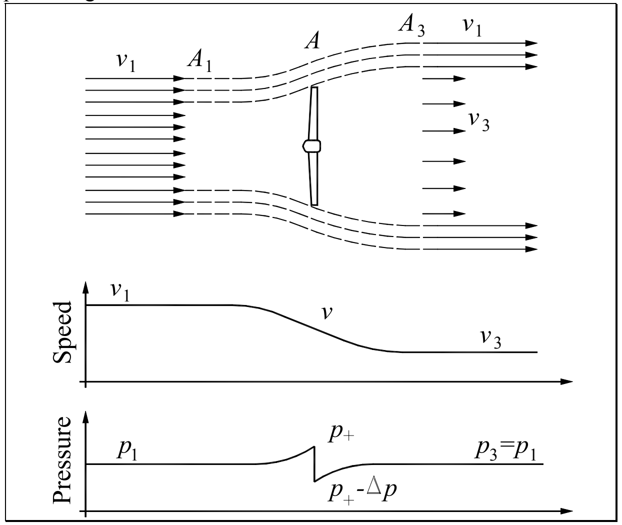


Figure 2.1: Interaction between wind and wind turbine

Assumption: The pressure changes are relatively small compared to the pressure in the ambient (about 1 atm = 101325 Pa) therefore we assume the density to be constant.

If we use (2.1) for the flow up-stream of the rotor, we get

$$p_1 + \frac{1}{2}\rho v_1^2 = p_+ + \frac{1}{2}\rho\ v^2 \quad \text{[Pa]} \quad (2.2)$$

If we use (2.1) down stream of the rotor plane, we get

$$p_+ - \Delta p + \frac{1}{2}\rho v^2 = p_1 + \frac{1}{2}\rho\ v_3^2 \quad \text{[Pa]} \quad (2.3)$$

Subtracting (2.3) from (2.2) we get

$$\Delta p = \frac{1}{2}\rho\left(v_1^2 - v_3^2\right) \quad \text{[Pa]} \quad (2.4)$$

Change of momentum: This differential pressure can also be calculated on the basis of"change in momentum". (For information see Appendix A). If we look at one square meter of the rotor plane, the mass flow equals $\rho\, v$. Momentum equals mass times velocity, with the unit N. Pressure equals force per surface, then the differential pressure can be calculated as

$$\Delta p = \rho\, v\left(v_1 - v_3\right) \quad \text{[Pa]} \quad (2.5)$$

Now (2.4) and (2.5) give

$$v = \frac{1}{2}\left(v_1 + v_3\right) \quad \text{[m/s]} \quad (2.6)$$

This indicates that the speed of air in the rotor plane equals the mean value of the speed upstream and down stream of the rotor.

Power production: The power of the turbine equals the change in kinetic energy in the air

$$P = \frac{1}{2}\rho\, v\left(v_1^{\,2} - v_3^{\,2}\right)A \quad \text{[W]} \quad (2.7)$$

Here A is the surface area swept by the rotor.

The axial force (thrust) on the rotor can be calculated as

$$T = \Delta p\, A \quad \text{[N]} \quad (2.8)$$

We now define ”the axial interference factor” a such that

$$v = \left(1 - a\right) v_1 \quad \text{[m/s]} \quad (2.9)$$

Using (2.6) and (2.9) we get $v_3 = (1 - 2a)\, v_1$ and (2.7) and (2.8) can be written as

$$P = 2\rho\, a\left(1 - a\right)^2 v_1^{\,3} A \quad \text{[W]} \quad (2.10)$$

$$T = 2\rho\, a\left(1 - a\right) v_1^{\,2} A \quad \text{[N]} \quad (2.11)$$

We now define two coefficients, one of the power production and one of the axial forces as

$$C_P = 4a\left(1 - a\right)^2 \quad \text{[-]} \quad (2.12)$$

$$C_T = 4a\left(1 - a\right) \quad \text{[-]} \quad (2.13)$$

Then (2.10) and (2.11) can be written as

$$P = \frac{1}{2}\rho\, v_1^{\,3} A\, C_P \quad \text{[W]} \quad (2.14)$$

$$T = ½\rho\, v_1^{\,2} A\, C_T \quad \text{[N]} \quad (2.15)$$

In figure 2.2, curves for C_P and C_T are shown.

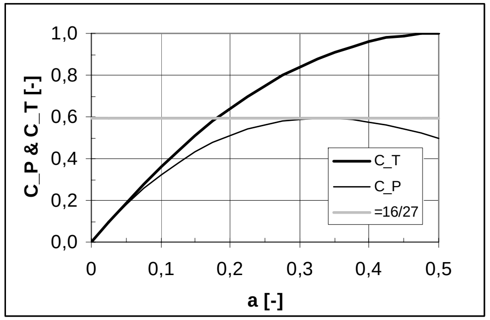


Figure 2.2: Coefficient of power C_P and coefficient of axial force C_T for an idealized wind turbine.

As shown, C_P has an optimum at about 0,593 (exactly 16/27) at an axial interference factor of 0,333 (exactly 1/3). According to Betz we have

$$P_{Betz} = C_{p,Betz}\, ½\rho\, v_1^{\,3} A \quad \text{with} \quad C_{P,Betz} = \frac{16}{27} \quad \text{[W]} \quad (2.16)$$

Example 2.1
Let us compare the axial force on rotor to the drag force on a flat plate? If a = 1/3 the C_T = 8/9 ≈ 0,89. Wind passing a flat plate with the area A would give a drag on the plate of

$$F_D = C_D \frac{1}{2}\rho\, v_1^{\,2} A \quad \text{[N]} \quad (2.17)$$

where C_D ≈ 1,1 i.e. the axial force on at rotor – at maximal power – is about 0,89/1,1 = 0,80 = 80% of the force on a flat plate of the same area as the rotor!

3. Rotor design

3.1. Air foil theory – an introduction

Figure 3.1 shows a typical wing section of the blade.

The air hits the blade in an angle α_A which is called the “angle of attack”. The reference line” for the angle on the blade is most often “the chord line” – see more in Chap. 4 for blade data. The force on the blade F can be divided into two components – the lift force F_L and the drag force F_D and the lift force is – per definition – perpendicular to the wind direction.

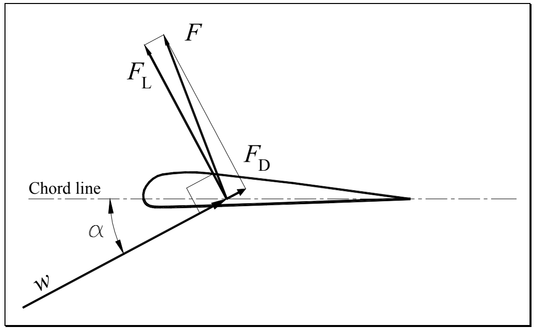


Figure 3.1: Definition of angle of attack

The lift force can be calculated as

$$F_{\mathrm{L}} = C_{\mathrm{L}} \frac{1}{2} \rho w^2 (bc) \qquad (3.1)$$

where C_L is the "coefficient of lift", ρ is the density of air, *w* the relative wind speed, *b* the width of the blade section and *c* the length of the chord line.

Similar for the drag force

$$F_{\mathrm{D}} = C_{\mathrm{D}} \frac{1}{2} \rho w^2 (bc) \qquad (3.2)$$

The coefficient of lift and drag both depend of the angle of attack, see figure 3.2.

For angles of attack higher than typically 15-20° the air is no longer attached to the blade, a phenomenon called "stall".

The ratio C_L/C_D is called the "glide ratio", i.e. $GR = C_L/C_D$. Normally we are interested in at high glide ratio for wind turbines as well as for air planes. Values up to 100 or higher is not uncommon and the angles of attack giving maximum are typical in the range 5 – 10°.

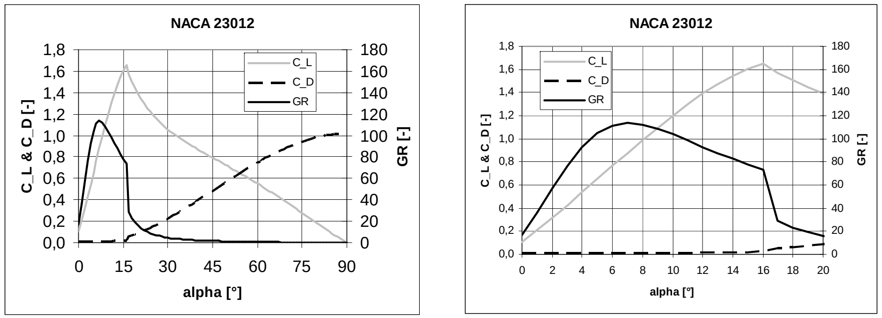


Figure 3.2: Coefficient of lift and drag as a function of the angle of attack (left: 0<α<90°; right: 0<α<20°)

3.2. Pitch angle, β, and chord length, c, after Betz

Figure 3.3 shows the velocities and the angles in a given distance, *r*, from the rotor axis. The rotor shown on the figure is with two blades, i.e. $B = 2$. To design the rotor we have to define the pitch angle β and the chord length *c*. Both of them depend on the given radius, that we are looking at therefore we sometimes write $\beta(r)$ and $c(r)$.

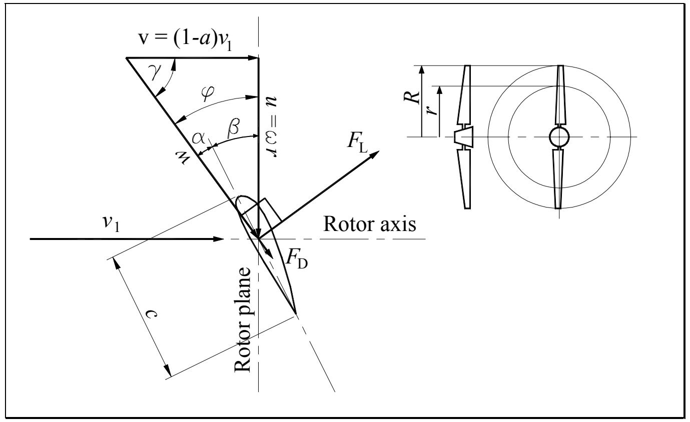


Figure 3.3. Velocities and angles

Angles, that all depends on the given radius

- $\gamma(r)$ = angle of relative wind to rotor axis
- $\varphi(r)$ = angle of relative wind to rotor plane
- $\beta(r)$ = pitch angle of the blade

The blade, as shown on the figure is moving up wards, thus the wind speed, seen from the blade, is moving down wards with a speed of *u*. We have

$$w^2 = v^2 + u^2 \qquad (3.3)$$

Betz does not include rotation of the wind, i.e. $a' = 0$ (see definition of a' later – formula (3.23)). Therefore

$$u = \omega r \qquad \text{[m/s]} \qquad (3.4)$$

Here ω is the angular speed of the rotor given by

$$\omega = 2\pi n \qquad \text{[rad/s]} \qquad (3.5)$$

where *n* is the rotational speed of the rotor in round per second.

Now we define the "tip speed ration" i.e.

$$X = \frac{v_{\mathrm{tip}}}{v_1} = \frac{\omega R}{v_1} \qquad [-] \qquad (3.6)$$

Combining these equations we get

$$\gamma(r) = \arctan \frac{3 r X}{2 R} \qquad [\mathrm{rad}] \qquad (3.7)$$

or

$$\varphi(r) = \arctan \frac{2 R}{3 r X} \qquad [\mathrm{rad}] \qquad (3.8)$$

and then the pitch angle

$$\boxed{\beta(r)_{\mathrm{Betz}} = \arctan \frac{2 R}{3 r X} - \alpha_{\mathrm{D}}} \qquad [\mathrm{rad}] \qquad (3.9)$$

where α_{D} is the angle of attack, used for the design of the blade. Most often the angle is chosen to be close to the angle, that gives maximum glide ration, see figure 3.2 that means in the range from 5 to 10°, but near the tip of the blade the angle is sometimes reduced.

Chord length, c(r):

If we look at one blade element in the distance *r* from the rotor axis with the thickness *dr* the lift force is, see formula (3.1) and (3.2)

$$\mathrm{d}F_{\mathrm{L}} = \frac{1}{2} \rho w^2 c \, \mathrm{d}r \, C_{\mathrm{L}} \qquad [\mathrm{N}] \qquad (3.10)$$

and the drag force

$$\mathrm{d}F_{\mathrm{D}} = \frac{1}{2} \rho w^2 c \, \mathrm{d}r \, C_{\mathrm{D}} \qquad [\mathrm{N}] \qquad (3.11)$$

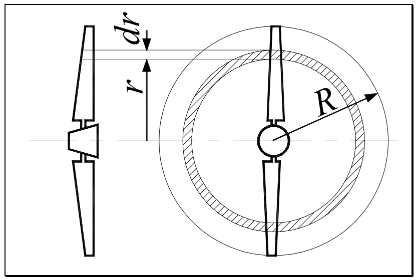


Figure 3.4: Blade section

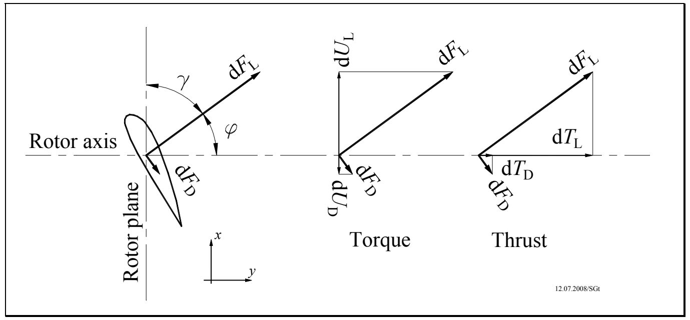


Figure 3.5. Forces on the blade element decomposed on the rotor plane, dU (torque), and in the rotor axis, d*T (thrust)*

For the rotor plane (torque) we have

$$\mathrm{d}U = \frac{1}{2} \rho w^2 c \, \mathrm{d}r \, C_{\mathrm{x}} \qquad [\mathrm{N}] \qquad (3.12)$$

with

$$C_{\mathrm{x}} = C_{\mathrm{L}} \sin(\varphi) - C_{\mathrm{D}} \cos(\varphi) \qquad [-] \qquad (3.13)$$

For the rotor axis (thrust) we have

$$\mathrm{d}T = \frac{1}{2} \rho w^2 c \, \mathrm{d}r \, C_{\mathrm{y}} \qquad [\mathrm{N}] \qquad (3.14)$$

with

$$C_{\mathrm{y}} = C_{\mathrm{L}} \cos(\varphi) + C_{\mathrm{D}} \sin(\varphi) \qquad [-] \qquad (3.15)$$

Now, in the design situation, we have $C_{\mathrm{L}} >> C_{\mathrm{D}}$, then (3.12) and (3.13) becomes

$$\mathrm{d}U = \frac{1}{2} \rho w^2 c \, \mathrm{d}r \, C_{\mathrm{L}} \cos(\gamma) \qquad [\mathrm{N}] \qquad (3.16)$$

and then the power produced

$$\mathrm{d}P = \mathrm{d}U \, r \, \omega \qquad [\mathrm{W}] \qquad (3.17)$$

If we have *B* blades, (3.16) including (3.17) gives

$$\mathrm{d}P = B \frac{1}{2} \rho w^2 c \, \mathrm{d}r \, C_{\mathrm{L}} \cos(\gamma) r \omega \qquad [\mathrm{W}] \qquad (3.18)$$

According to Betz, the blade element would also give

$$dP = \frac{16}{27}\frac{1}{2}\rho v_1^3 (2\pi r\,dr) \qquad [W] \qquad (3.19)$$

Using $v_1 = 3/2\, w \cos(\gamma)$ and $u = w \sin(\gamma)$, then (3.18) and (3.19) gives

$$c(r)_{\text{Betz}} = \frac{16\pi R}{9 B C_{L,D}} \frac{1}{X\sqrt{X^2\left(\frac{r}{R}\right)^2 + \frac{4}{9}}} \qquad [m] \qquad (3.20)$$

where $C_{L,D}$ is the coefficient of lift at the chosen design angle of attack, $\alpha_{A,D.}$

Example 3.1

What will be shown later is that a tip speed ration of about $X = 7$ is optimal (see fig. 6.2). Further more 3 blades seem to be state of the art. Figure 3.6 and 3.7 shows the results of formula (3.20) concerning the chord length i.e. according to Betz.

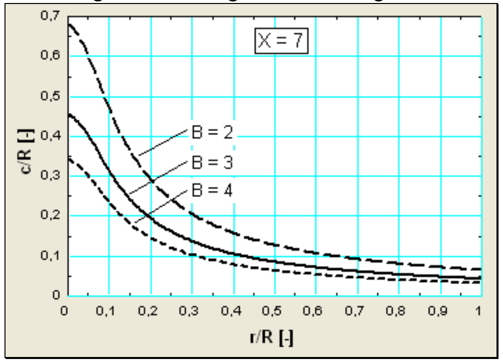


Figure 3.6. Chord length as function of radius for X = 7 and for different numbers of blades

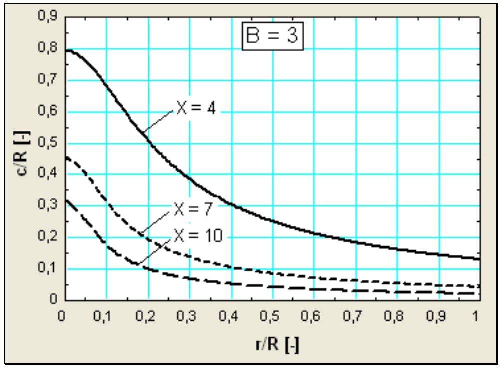


Figure 3.7. Chord length as function of radius for three blades B = 3 and for different tip speed ratios

3.3. Pitch angle, β, and chord length, c, after Schmitz

Schmitz has developed a little more detailed and sophisticated model of the flow in the rotor plane. The torque M in the rotor shaft can only be established because of the rotation of the wake, cf. Appendix A which is a result of the conservation law for angular momentum

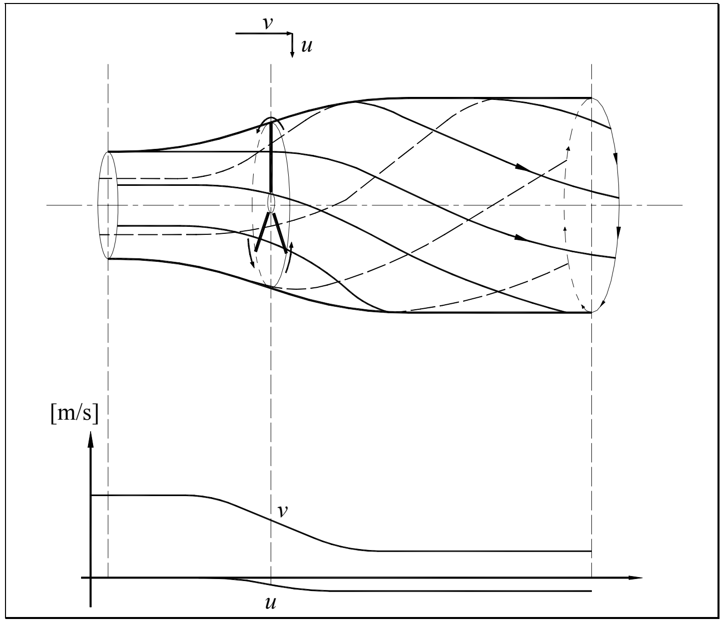


Figure 3.8. Down stream rotation of the wake – The wake rotates in the opposite direction to the rotor

The power can be calculated as

$$P = M\,\omega \qquad [W] \qquad (3.21)$$

where M is the torque in the rotor shaft and ω is the angular speed. According to the conservation rule of angular momentum, the torque in the rotor shaft can only be established because of a swirl induced in the slipstream in the flow down stream of the rotor. As for the axial speed v it can be shown theoretically that the change in the tangential speed in the rotor plane is half of the total change, i.e. we have in the rotor plane

$$u = r\,\omega + ½\,\Delta u \qquad [m/s] \qquad (3.22)$$

or

$$u = r\,\omega(1 + a') \quad \text{[m/s]} \quad (3.23)$$

which defines the “tangential interference factor a’ “

As mentioned previously index 1 is used for the upstream situation, index 2 and 3 for rotor plane and downstream respectively. In the following index 2 is some times omitted – for simplicity.

Now look at the flow in the rotor plane, see figure 3.9. What is important here is the relation

$$\vec{w} = \vec{w}_1 + ½\,\Delta\vec{w} \quad \text{[m/s]} \quad (3.24)$$

The change in w_1 is because of the air foil effect. If we assume that the drag is very low (compared to lift, i.e. $C_D << C_L$ => $C_D \approx 0$) then the Δw vector is parallel to the lift force vector dF_L (because of the conservation law of momentum) and we – per definition of the direction of lift force – also have that the Δw vector is perpendicular to w – see figure 3.9-b4). Based on these considerations we have the following geometrical relations

$$w = w_1 \cos(\varphi_1 - \varphi) \quad \text{[m/s]} \quad (3.25)$$

and from figure 3.9. – b2)

$$v = w \sin(\varphi) \quad \text{[m/s]} \quad (3.26)$$

Combining (3.25) and (3.26) we get

$$v = w_1 \cos(\varphi_1 - \varphi)\sin(\varphi) \quad \text{[m/s]} \quad (3.27)$$

From figure 3.9 we further have

$$\Delta w = 2 w_1 \sin(\varphi_1 - \varphi) \quad \text{[m/s]} \quad (3.28)$$

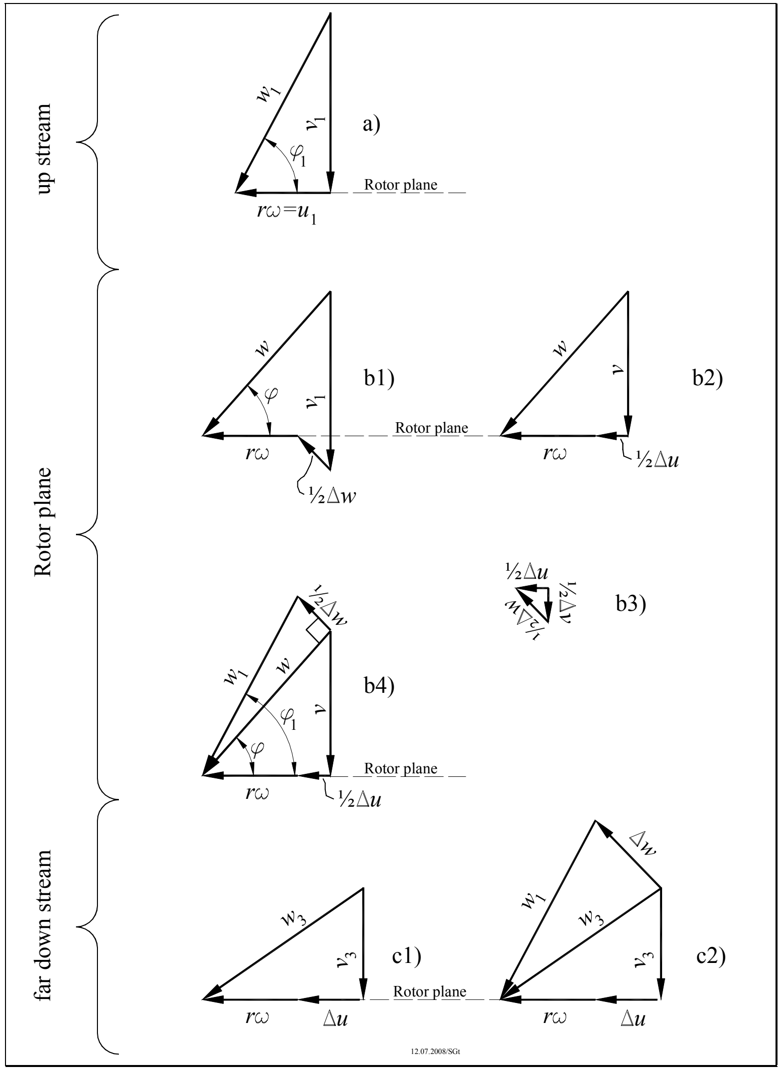


Figure 3.9. Speed in the rotor plane a) far upstream; b) in the rotor plane and; c) far down stream

Now, let us look at the power! From the conservation of momentum we have

$$\mathrm{d}F_L = \Delta w\,\mathrm{d}q \quad \text{[N]} \quad (3.29)$$

where dq is the mass flow through the ring element in the radius r with the width dr, i.e.

$$\mathrm{d}q = 2\rho\,\pi\, r\,\mathrm{d}r\, v \qquad \text{[kg/s]} \qquad (3.30)$$

Power equals "torque multiplied by angular velocity" and (neglecting drag) then

$$\begin{aligned}
\mathrm{d}P &= \mathrm{d}M\,\omega \\
&= \mathrm{d}F_{\mathrm{L}}\,\sin(\varphi)\, r\,\omega \\
&= \Delta w\,\mathrm{d}q\,\sin(\varphi)\, r\,\omega \qquad \text{[kg/s]} \qquad (3.31)\\
&= \{2\,w_1 \sin(\varphi_1-\varphi)\}[(2\rho\,\pi\, r\,\mathrm{d}r)\, w_1 \cos(\varphi_1-\varphi)\sin(\varphi)]\sin(\varphi) r\omega \\
&= r^2\omega\,\rho\,2\pi\,\mathrm{d}r\, w_1^2 \sin[2(\varphi_1-\varphi)]\sin^2(\varphi_1)
\end{aligned}$$

In the bottom transaction above we have used the relation sin(x) cos(x) = sin(2x).

We have now a relation for the power of the ring element as a function of the angle φ but we do not know this angle? The trick is now to solve the equation d(dP)/dφ = 0 to find the angle that gives maximum power. Doing this for (3.31) we get

$$\begin{aligned}
\frac{\mathrm{d}(\mathrm{d}P)}{\mathrm{d}\varphi} &= (r^2\omega\,\rho\,2\pi\,\mathrm{d}r\, w_1^2)(-2\cos[2(\varphi_1-\varphi)]\sin^2\varphi + 2\sin[2(\varphi_1-\varphi)]\sin\varphi\cos\varphi) \\
&= (r^2\omega\,\rho\,2\pi\,\mathrm{d}r\, w_1^2)2\sin\{\sin[2(\varphi_1-\varphi)]\cos\varphi - \cos[2(\varphi_1-\varphi)]\sin\varphi\} \qquad \text{[W/°]} \quad (3.32)\\
&= (r^2\omega\,\rho\,2\pi\,\mathrm{d}r\, w_1^2)2\sin\varphi\{\sin(2\varphi_1 - 3\varphi)\}
\end{aligned}$$

From d(dP)/dφ = 0, it follows

$$\boxed{\varphi_{\max} = \frac{2}{3}\varphi_1} \qquad \text{[rad]} \qquad (3.33)$$

or

$$\varphi_{\max} = \frac{2}{3}\arctan\frac{v_1}{\omega r} = \frac{2}{3}\arctan\frac{R}{X\,r} \qquad \text{[rad]} \qquad (3.34)$$

and the for pitch angle

$$\boxed{\beta(r)_{\mathrm{Schmitz}} = \frac{2}{3}\arctan\frac{R}{r\,X} - \alpha_{\mathrm{D}}} \qquad \text{[rad]} \qquad (3.35)$$

Example 3.2

Let's compare Betz' and Schmitz' formulas for the design of the optimal pitch angle. Assuming X = 7; B = 3; α_D =7,0°; C_L = 0,88 one gets

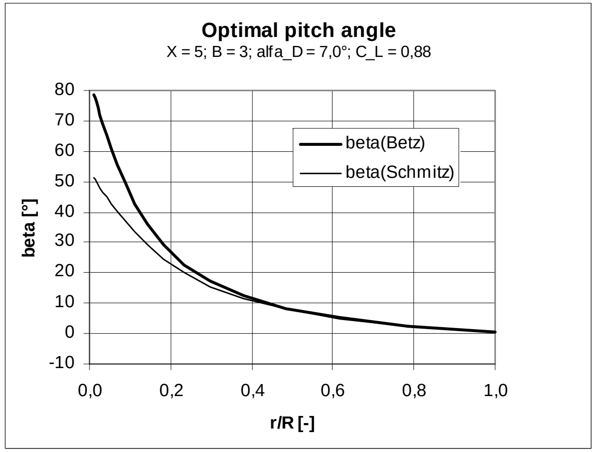


Figure 3.10: Optimal pitch angel

Note, that only for small r/R the two theories differ. And here the power produced is small because of the relatively small swept area. At the tip (r/R = 1) the optimal angle is approx. 0,5° for both.

Using the result of (3.27), (3.28) and (3.33) in (3.29) we get

$$\begin{aligned}
\mathrm{d}F_{\mathrm{L}} &= \Delta w\,\mathrm{d}q \\
&= 2\,w_1\sin(\varphi_1-\varphi)2\rho\pi\, r\,\mathrm{d}r(w_1\cos(\varphi_1-\varphi)\sin(\varphi)) \\
&= 2\,w_1^{\,2}\,2\rho\pi\, r\,\mathrm{d}r\sin\left(\frac{\varphi_1}{3}\right)\cos\left(\frac{\varphi_1}{3}\right)\sin\left(\frac{2\varphi_1}{3}\right) \qquad \text{[N]} \qquad (3.36)\\
&= 2\,w_1^{\,2}\,2\rho\pi\, r\,\mathrm{d}r\sin^2\left(\frac{\varphi_1}{3}\right)\cos^2\left(\frac{\varphi_1}{3}\right)
\end{aligned}$$

where we again use sin(2x) = 2 sin(x)cos(x).

From the air foil theory we have

$$\begin{aligned}
\mathrm{d}F_{\mathrm{L}} &= ½\rho\, w^2 B\, c\,\mathrm{d}r\, C_{\mathrm{L}} \\
&= ½\rho\, w_1^{\,2} B\, c\,\mathrm{d}r\, C_{\mathrm{L}}\cos\left(\frac{\varphi_1}{3}\right) \qquad [7] \qquad (3.37)
\end{aligned}$$

where we have used (3.25) and $\varphi = 2/3\varphi_1$.

Combining (3.37) and (3.36) we get

$$c(r)_{\mathrm{Schmitz}} = \frac{1}{B}\frac{16\pi\, r}{C_{\mathrm{L}}}\sin^2\left(\frac{\varphi_1}{3}\right) \qquad \text{[m]} \qquad (3.38)$$

or

$$c(r)_{\text{Schmitz}} = \frac{1}{B}\frac{16\pi r}{C_L}\sin^2\left(\frac{1}{3}\arctan\left(\frac{R}{X r}\right)\right) \quad \text{[m]} \quad (3.39)$$

Example 3.3

Let's again compare Betz' and Schmitz' formulas for the design of the optimal pitch angle. Assuming $X = 7$; $B = 3$; $\alpha_D = 7{,}0°$; $C_L = 0{,}88$ one gets

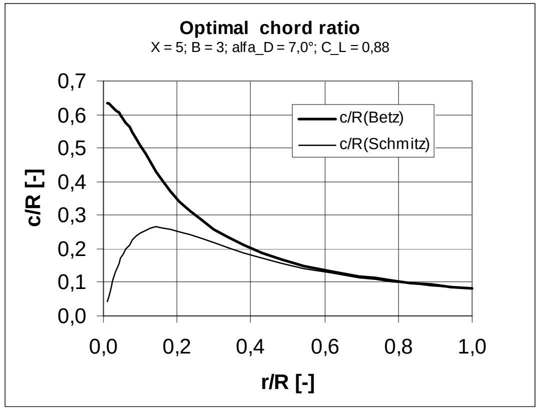


Figure 3.11: Optimal chord length

Note, near the tip there are no difference between Betz' and Schmitz' theory.

4. Characteristics of rotor blades

Wing profiles are often tested in wind tunnels. Results are curves for coefficient of lift and drag and moment. Data for a lot of profiles can be found in "Theory of Wing Sections, Ira H. Abbott and A. E. Doenhoff, ref./3/.

Figure 4.1 shows data for the profile NACA 23012.

Lift, drag and torque (per meter blade width) are defined by the equations

$$F_L^* = ½\rho w^2 c C_L \quad \text{[N]} \quad (4.1)$$

$$F_D^* = ½\rho w^2 c C_D \quad \text{[N]} \quad (4.2)$$

$$Q_M^* = ½\rho w^2 c^2 C_M \quad \text{[Nm]} \quad (4.3)$$

The density of air is at a nominal state, defined as 1 bar and 11°C, 1,225 kg/m^3.

The curves in figure 4.1 are given at different Reynolds's number, defined as

$$\text{Re} = \frac{c\, w}{\mu / \rho} \quad \text{[-]} \quad (4.4)$$

For PC-calculation it is convenient to have the curves as functions. For the NACA 23012 profile one can use the following approximation: $C_{D,L} = k_0 + k_1\alpha + k_2\alpha^2 + k_3\alpha^3 + k_4\alpha^4$, with the following constants

NACA 23012		
	C_L	C_D
k_0	1,0318e-1	6,0387e-3
k_1	1,0516e-1	–3,6282e-4
k_2	1,0483e-3	5,4269e-5
k_3	7,3487e-6	6,5341e-6
k_4	–6,5827e-6	–2,8045e-7

Table 4.1: Polynomial constants – for 0 < α < 16°

As shown in figure 4.2, the data are given in the range of $\alpha < 20°$. For wind turbines it is necessary to know the data for the range up to 90°. In the range from $\alpha_{st} < \alpha < 90°$ we can use the following assumptions, see ref./2/

Lift:

$$C_L = A_1 \sin(2\alpha) + A_2 \frac{\cos^2(\alpha)}{\sin(\alpha)} \quad \text{[-]} \quad (4.5)$$

where

$$A_1 = \frac{B_1}{2} \quad (B_1 \ \text{see (4.8) below!})$$

$$A_2 = \left(C_{Ls} - C_{D,\max}\sin(\alpha_{st})\cos(\alpha_{st})\right)\frac{\sin(\alpha_{st})}{\cos^2(\alpha_{st})} \quad \text{[-]} \quad (4.6)$$

Drag:

$$C_D = B_1 \sin^2(\alpha) + B_2\cos(\alpha) + C_{Ds} \quad \text{[-]} \quad (4.7)$$

where C_{Ds} is the coefficient of drag at the beginning of stall, α_{stall}, and

$$B_1 = C_{D,\max}$$

$$B_2 = \frac{1}{\cos(\alpha_{st})}\left(C_{Ds} - C_{D,\max}\sin^2(\alpha_{st})\right) \quad \text{[-]} \quad (4.8)$$

C_{Dmax} can be set at 1. For the NACA 23012 profile, the angle of stall is a little uncertain, but could in practice be set at 16°. Figure 3.2 show the result of the formulas above.

Figure 4.1 show some typical data for an air foil.

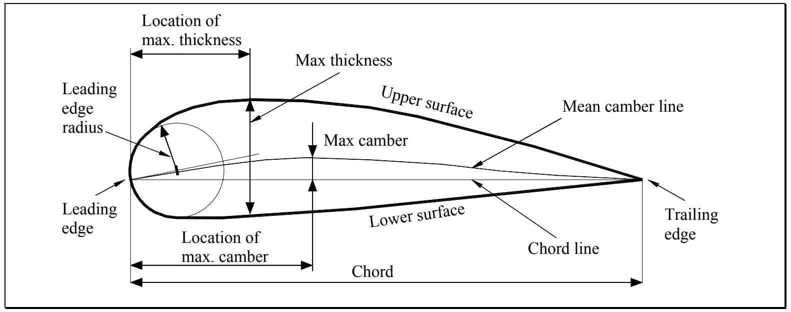
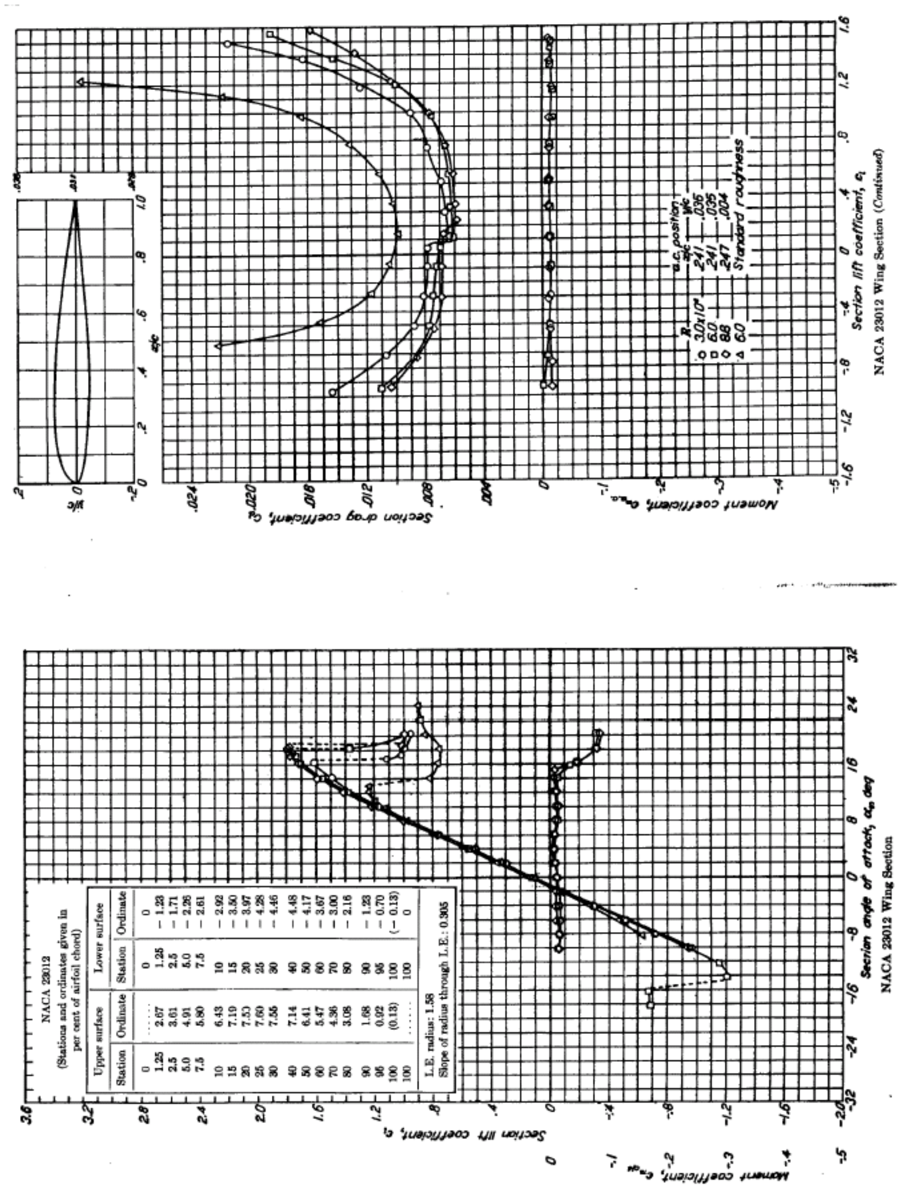


Figure 4.1: Definition of typical air foil data

- The *chord line* is a straight line connecting the leading and the trailing edges of the air foil.
- The *mean camber line* is a line drawn halfway between the upper and the lower surfaces. The chord line connects the ends of the mean camber lines.
- The frontal surface of the airfoil is defined by the shape of a circle with the *leading edge radius (L.E. radius)*.
- The center of the circle is defined by the leading edge radius and a line with a given *slope of the leading edge radius* relative to the chord.

Data for the NACA 23012 profile is given by the table (upper left corner) on figure 4.2.



Upper surface Station	Upper surface Ordinate	Lower surface Station	Lower surface Ordinate
0	.....	0	0
1.25	2.67	1.25	− 1.23
2.5	3.61	2.5	− 1.71
5.0	4.91	5.0	− 2.26
7.5	5.80	7.5	− 2.61
10	6.43	10	− 2.92
15	7.19	15	− 3.50
20	7.50	20	− 3.97
25	7.60	25	− 4.28
30	7.55	30	− 4.46
40	7.14	40	− 4.48
50	6.41	50	− 4.17
60	5.47	60	− 3.67
70	4.36	70	− 3.00
80	3.08	80	− 2.16
90	1.68	90	− 1.23
95	0.92	95	− 0.70
100	(0.13)	100	(− 0.13)
100	.....	100	0

L.E. radius: 1.58
Slope of radius through L.E.: 0.305

Figure 4.2: Data for NACA 23012 (Ref./3/)

5. The blade element momentum (BEM) theory

In the blade element momentum (BEM) method the flow area swept by the rotor is divided into a number of concentric ring elements. The rings are considered separately under the assumption that there is no radial interference between the flows in one ring to the two neighbouring rings.

Figure 3.3 shows the profile and the wind speeds in one ring. The angle of attack α is given by

$$\alpha = \varphi - \beta \quad \text{[rad]} \quad (5.1)$$

From figure 3.3 we get

$$\tan(\varphi) = \frac{1-a}{1+a'} \frac{v_1}{r\,\omega} \quad \text{[-]} \quad (5.2)$$

If the number of blades is *B,* we can calculate the axial force d*T* and the torque d*U* on a ring element with the radius *r* and the width d*r* and the torque as

$$\mathrm{d}T = ½\,\rho\, w^2 c\, B\, C_y \mathrm{d}r \quad \text{[N]} \quad (5.3)$$

$$\mathrm{d}U = ½\,\rho\, w^2 c\, B\, C_x\, r\, \mathrm{d}r \quad \text{[Nm]} \quad (5.4)$$

where C_y and C_x are given by (3.15) and (3.13)

If we now use the laws of momentum and angular momentum, we get

$$\mathrm{d}T = 2\pi\, r\, \rho\, v_2 (v_1 - v_3)\mathrm{d}r \quad \text{[N]} \quad (5.5)$$

$$\mathrm{d}U = 2\pi\, r^2 \rho\, v_2\, u_3\, \mathrm{d}r \quad \text{[Nm]} \quad (5.6)$$

In (5.6) we are using u_3 for the tangential speed far behind the rotor plane, even though there is some tangential rotation of the wind. This can be shown to be an allowable approximation, because the rotation of the wind normally is small.

Combining (5.3) and (5.5) - and - (5.4) and (5.6) we get

$$\frac{a}{a-1} = \frac{c\, B\, C_y}{8\pi\, \mathrm{r} \sin^2(\Phi)} \quad \text{[-]} \quad (5.7)$$

$$\frac{a'}{a'+1} = \frac{c\, B\, C_x}{8\pi\, \mathrm{r} \sin(\Phi)\cos(\Phi)} \quad \text{[-]} \quad (5.8)$$

Here we have used

$$w = \frac{v_1(1-a)}{\sin(\varphi)} \quad \text{[m/s]} \quad (5.9)$$

or

$$w = \frac{\omega r(1+a')}{\cos(\varphi)} \quad \text{[m/s]} \quad (5.10)$$

If we now define the solid ratio as

$$\sigma = \frac{c\, B}{2\pi\, r} \quad \text{[-]} \quad (5.11)$$

and solve the equation (5.7) and (5.9) we get

$$a = \frac{1}{\frac{4\sin^2(\varphi)}{\sigma\, C_y} + 1} \quad \text{[-]} \quad (5.12)$$

and

$$a' = \frac{1}{\frac{4\sin(\varphi)\cos(\varphi)}{\sigma\, C_x} - 1} \quad \text{[-]} \quad (5.13)$$

For rotors with few blades it can be shown that a better approximation of *a* and *a'* is

$$a = \frac{1}{\frac{4F\sin^2(\varphi)}{\sigma\, C_y} + 1} \quad \text{[-]} \quad (5.14)$$

and

$$a' = \frac{1}{\frac{4F\sin(\varphi)\cos(\varphi)}{\sigma\, C_x} - 1} \quad \text{[-]} \quad (5.15)$$

where

$$F = \frac{2}{\pi} \arccos\left(\exp\left(-\frac{B}{2} \frac{R-r}{r\sin(\varphi)} \right) \right) \quad \text{[N]} \quad (5.16)$$

This simple momentum theory breaks down when a becomes greater than a_c = 0,2. In that case we replace (5.14) by

$$a = ½\left(2 + K(1-2a_c) - \sqrt{(K(1-2a_c)+2)^2 + 4(K a_c^2 - 1)}\right) \quad [-] \quad (5.17)$$

where

$$K = \frac{4F\sin^2(\varphi)}{\sigma C_y} \quad [-] \quad (5.18)$$

Calculation procedure

We can now calculate the axial force and power of one ring element of the rotor by making the following iteration:

For every radius r (4 to 8 elements are OK), go through step-1 to step-8

Step-1: Start

Step-2: a and a' are set at some guessed values. $a = a' = 0$ is a good first time guess.

Step-3: φ is calculated from (5.2)

Step-4: From the blade profile data sheet (or the polynomial approximation) we find C_L and C_D

Step-5: C_x and C_y are calculated by (3.13) and (3.15)

Step-6: a and a' are calculated by (5.14) and (5.15). Or if $a > 0,2$ then a is calculated from (5.17).

Step-7: If a and a' as found under step-5 differ more than 1% from the last/initial guess, continue at step-2, using the new a and a'.

Step-8: Stop

When the iterative process is ended for all blade elements, then the axial force and tangential force (per meter of blade) for any radius can be calculated as

$$U^*(r) = ½\rho w^2 c C_x \quad [N] \quad (5.19)$$

$$T^*(r) = ½\rho w^2 c C_y \quad [N] \quad (5.20)$$

and then the total axial force and power as

$$T = B\int_0^R T^*(r)\mathrm{d}r \quad [N] \quad (5.21)$$

$$P = \omega B\int_0^R r U^*(r)\mathrm{d}r \quad [N] \quad (5.22)$$

6. Efficiency of the wind turbine

6.1. Rotor

Betz has shown that the maximum power available in the wind is given by (2.16). Let us define this power as

$$P_{max} = \frac{16}{27}\frac{1}{2}\rho v_1^3 A \quad [W] \quad (6.1)$$

where we have used $C_p = C_{p,Betz}$ =16/27.

In (6.1) A is the swept area of the rotor, and in the following we define this area as $A = \pi/4\ D^2$ i.e. we do not take into account, that some part of the hub area is not producing any power!

We can now define the rotor efficiency as

$$\eta_{rotor} = \frac{P_{rotor}}{P_{max}} \quad [-] \quad (6.2)$$

where P_{rotor} is the power in the rotor shaft.

The rotor efficiency can be calculated on the basis of a BEM-calculation of the power production in a real turbine – see the example in Chapter 7.

Another model will be presented here:

The rotor efficiency is divided into three parts

$$\eta_{rotor} = \eta_{wake}\eta_{tip}\eta_{profile} \quad [-] \quad (6.3)$$

where “wake” indicates the loss because of rotation of the wake, “tip” the tip loss and “profile” the profile losses.

Wake loss:

The wake loss can be calculated on the basis of Schmitz’ theory. Integrating (3.31) over the whole blade area and using (3.8) and (3.33) gives.

$$P_{\text{Schmitz}} = \tfrac{1}{2}\rho \frac{\pi}{4} D^2 v_1^{\,3} \int_0^1 4\,X \left(\frac{r}{R}\right)^2 \frac{\sin^3\!\left(\frac{2}{3}\varphi_1\right)}{\sin^2(\varphi_1)} \,\mathrm{d}\!\left(\frac{\mathrm{r}}{\mathrm{R}}\right) \quad \text{[W]} \qquad (6.4)$$

This can be solved numerically, see an example in Appendix D. Based on this we can define

$$C_{\text{p,Schmitz}} = \frac{P_{\text{Schmitz}}}{\frac{1}{2}\rho v_1^{\,3} A} \quad \text{[-]} \qquad (6.5)$$

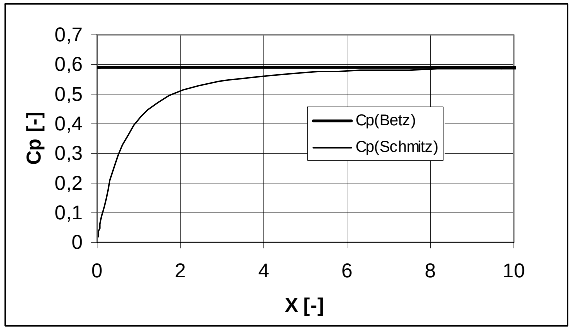


Figure 6.1. Coef. of power according to Betz and Schmitz

The difference between Betz and Schmitz is, that Schmitz takes the swirl loss into account and therefore we can define swirl loss or the wake loss as

$$\eta_{\text{wake}} = \frac{C_{\text{p,Schmitz}}}{C_{\text{p,Betz}}} \quad \text{[-]} \qquad (6.6)$$

Tip loss:

In operation there will be a high negative (compared to ambient) pressure above the blade and a (little) positive pressure under the blade. Near the tip of the blade, this pressure difference will induce a by pass flow from the high pressure side to low pressure side – over the tip end of the blade – thus reducing the differential pressure and then the power production!

The model of Betz – see ref. /4/, page 153-155 – results in a tip efficiency of

$$\eta_{\text{tip}} = \left(1 - \frac{0{,}92}{B\sqrt{X^2 + 4/9}}\right)^2 \quad \text{[-]} \qquad (6.7)$$

Profile loss:

From the power calculation after (3.12) and (3.13) we can see, that the power is proportional to C_x. For an ideal profile, i.e. with no drag, the power would the be higher, from which we can define the profile efficiency to

$$\eta_{\text{profile}}(r) = \frac{C_L \cos(\gamma) - C_D \sin(\gamma)}{C_L \cos(\gamma)} = 1 - \frac{C_D}{C_L}\tan(\gamma) \quad \text{[-]} \qquad (6.8)$$

Using (3.7) we get

$$\eta_{\text{profile}}(r) = 1 - \frac{3\,r\,X}{2\,R\;GR} \quad \text{[-]} \qquad (6.8)$$

Assuming the angle of attack to be the same over the entire blade length the glide ratio is constant too and then (6.8) can be integrated over the blade length to give

$$\eta_{\text{profile}} = 1 - \frac{X}{GR} \quad \text{[-]} \qquad (6.9)$$

Example 6.1

Assuming the glide ration to be GR = 100 and the blade number to B = 3 then the rotor efficiency can be calculated as function of the tip speed ratio, see figure 6.2.

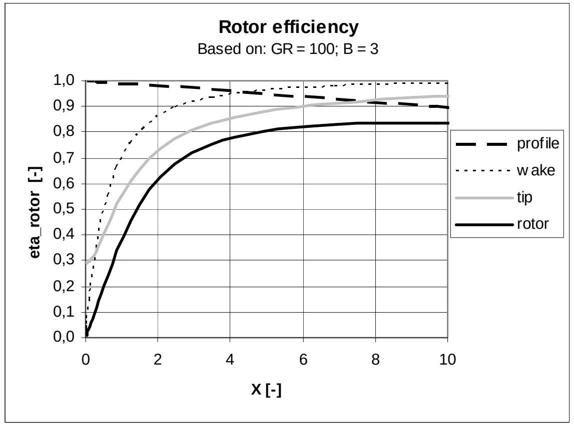


Figure 6.2. Rotor efficiency

Most modern wind turbines have tip speed ration at nominal wind speed and power around x = 7, and from the curve it is obvious, that this is close to optimal!

Example 6.2

Most modern wind turbines have glide ratios around 100 and three blades. Figure 6.3 shows the rotor efficiency for 2,3 and 4 blades and with the glide ratio as parameter.

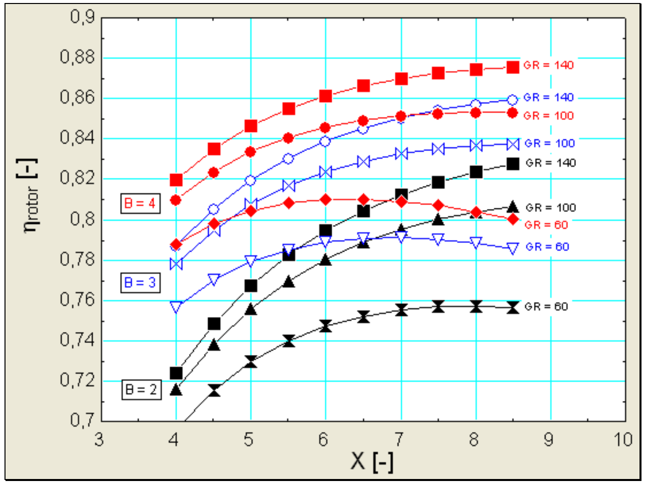


Figure 6.3. Rotor efficiency

For $X = 7$ and for a glide ratio $GR = 100$ it can bee seen, that the number of blades have the following influence on the rotor efficiency

2 blades: 79,5%
3 blades: 83,3%
4 blades: 85,1%

3 and 4 blades are more efficient than 2 blades, but also more expensive. When a 3 blade rotor in spite of that has become a de facto standard it is due to a more dynamical stable rotor.

6.2. Gear box, generator and converter

Most wind turbines have the following main parts, a rotor, a gear box a generator and an electric converter, see figure 6.4. Each of these components has losses.

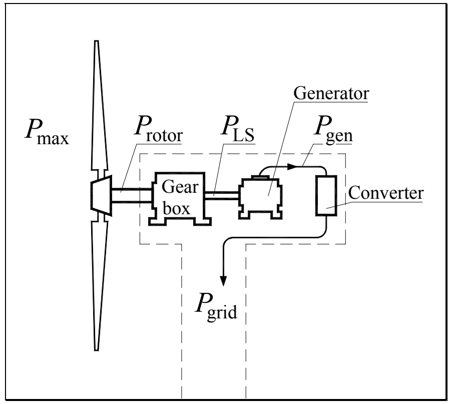


Figure 6.4: Main components in a wind turbine

The total efficiency of such a turbine can the be defined as

$$\eta_{total} = \frac{P_{grid}}{P_{max}} = \eta_{rotor}\eta_{gearbox}\eta_{gen}\eta_{conv} \qquad [-] \qquad (6.10)$$

where

$$\eta_{rotor} = \frac{P_{rotor}}{P_{max}}, \quad \eta_{gearbox} = \frac{P_{HS}}{P_{rotor}}, \quad \eta_{gen} = \frac{P_{gen}}{P_{HS}}, \quad \eta_{conv} = \frac{P_{grid}}{P_{gen}} \qquad [-] \qquad (6.11)$$

where the indices stand for “LS” = low speed (shaft); “gen” = generator; “conv” = frequency converter and “grid” = grid net.

Typical values for the efficiencies are – at nominal power

Gearbox: 0,95-0,98
Generator: 0,95-0,97
Converter: 0,96-0,98

At part load, the lower values can be expected.

Cooling:

The cooling of the components can be calculated as “power input minus power output”. As an example for the gear box: $\Phi_{gearbox} = P_{rotor} - P_{LS}$.

7. Example, BEM

The programming can be done in a spread sheet (Excel). Let us look at an example.

We would like to design a wind turbine with 3 blades, a nominal tip speed ratio of $X = 5$ and to use the NACA profile 23012. For this profile we have an optimum glide number ($=C_L/C_D$) at an angle of attack on 7,0° and here the coefficient of lift is 0,88. The result is shown in figure 7.1.

Next we want to calculate the power and axial force on the turbine with a radius of 5 m at a wind speed of 10 m/s and at a rotary speed of 88 rpm. The calculation is shown in figure 7.2.

Figure 7.3 shows the performance of the turbine at a fixed number of revolutions but varying wind speed.

A	B	C	D	E	F	G	H	I	J	K
2										
3	**Design procedure for an optimal rotor**									
4	2008.07.05/SGt						Blue numbers are input to the program			
5										
6							Outputs are teta and c/R, see figures			
7	Tip speed ratio	X	-	5						
8	Number of blades	B	-	3			8 element-model where there is no blade			
9	Angle of attack	alpha	°	7			at the inner element!			
10	Coeff. of lift	C_L	-	0,88						
11	Table 1: Input									
12										
13										
14	Rel. radius	r/R	-	0,188	0,313	0,438	0,563	0,688	0,813	0,938
15	Speed ratio	x	-	0,94	1,56	2,19	2,81	3,44	4,06	4,69
16	Angle, optimal	phi	°	31,2	21,7	16,4	13,0	10,8	9,2	8,0
17	Pitch	beta	°	24,2	14,7	9,4	6,0	3,8	2,2	1,0
18	Rel. chord length	c/R	-	0,259	0,212	0,169	0,138	0,116	0,100	0,087
19	Table 2: Output (according to Schmitz)									

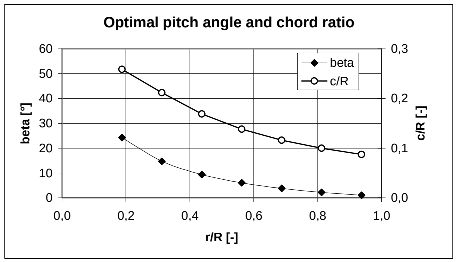


Figure 7.1: Design of the rotor (Formulas, see App. B)

	A	B	C	D	E	F	G	H	I	J
1	1	B	C	D	E	F	G	H	I	J
2	2	**Power calculation**								
3	3	2008.07.05/SGt								
4	4								**Wing performance**	
5	5								k0	0,10318
6	6	Radius	R_o	m	5				k1	0,10516
7	7	Wind speed	v_1	m/s	10				k2	0,001048
8	8	Rotational speed	n	min^-1	88				k3	7,35E-06
9	9	Density of air	rho	kg/m3	1,225				k4	-6,58E-06
10	10	Number of blades	B	-	3					
11	11	Angular speed	omega	s^-1	9,2				alfa_s	16
12	12	Thickness, 1 ring	dr	m	0,625				C_ls	1,652801
13	13	Inner radius	R_i	m	0,625				C_ds	2,25E-02
14	14	Swept surface	A_s	m2	78,54				B1	1,00
15	15	Max. power	P_max	W	28507				B2	-0,0556
16	16	Tip speed	V_TIP	m/s	46,1				A1	0,500
17	17	Tip speed ratio	X_act	-	4,61				A2	0,414
18	18	Table 1: Rotor size and speed							Table 2: Wing data	
19	19									
20	20									
21	21	Ring no.	N_r	-	1	2	3	4	5	6
22	22	Rel. radius	r/R	-	0,19	0,31	0,44	0,56	0,69	0,81
23	23	Radius	r	m	0,94	1,56	2,19	2,81	3,44	4,06
24	24	Pitch	beta	°	24,232	14,746	9,378	6,049	3,813	2,219
25	25	Chord	c	m	1,293	1,059	0,845	0,691	0,581	0,500
26	26	Table 3: From "Design"								
27	27				If you want to use the pitch and cord values from the optimal rotor design					
28	28				copy the values in rows 24 and 25 to rows 29 and 30!					
29	29	Pitch, no pitch control	beta_0	°	24,232	14,746	9,378	6,049	3,813	2,219
30	30	Chord	c	m	1,293	1,059	0,845	0,691	0,581	0,500
31	31	Pitch angle	beta	°	24,2	14,7	9,4	6,0	3,8	2,2
32	32	Solid ratio	sigma	-	0,66	0,32	0,18	0,12	0,08	0,06
33	33	Speed of blade	r*omega	m/s	8,6	14,4	20,2	25,9	31,7	37,4
34	34				Use the macro "Ctrl + p" to made the iteration ! You might have to use t					
35	35				more times to reduce the errors in rows 51 and 52 to 0					
36	36			-						
37	37	Axial int.factor	a	-	0,316	0,319	0,315	0,310	0,309	0,320
38	38	Tang. int. factor	a'	-	0,243	0,099	0,052	0,031	0,021	0,015
39	39	Angle of rel. wind	PHI	°	32,5	23,3	17,9	14,5	12,1	10,1
40	40	Angle of attack	alpha	°	8,2	8,5	8,5	8,4	8,3	7,9
41	41	Coeff. of lift	C_L	-	1,0154	1,0460	1,0461	1,0348	1,0159	0,9794
42	42	Coeff. of drag	C_d	-	0,0091	0,0095	0,0095	0,0093	0,0091	0,0087
43	43	y-component	C_y	-	0,861	0,965	0,998	1,004	0,995	0,966
44	44	x-component	C_x		0,538	0,405	0,313	0,250	0,203	0,164
45	45	Factor	F	-	1,000	1,000	0,999	0,994	0,976	0,911
46	46	Factor	K	-	2,032	2,001	2,051	2,107	2,122	1,990
47	47	Axial int.factor (1)	a_1	-	0,330	0,333	0,328	0,322	0,320	0,334
48	48	Axial int.factor (2)	a_2		0,316	0,319	0,315	0,310	0,309	0,320
49	49	Axial int.factor	a		0,316	0,319	0,315	0,310	0,309	0,320
50	50	Tang. int. factor	a'	-	0,243	0,099	0,052	0,031	0,021	0,015
51	51	a and a' must be	error_a	%	0	0	0	0	0	0
52	52	guessed until errors	error_a'	%	0	0	0	0	0	0
53	53	< 2%								
54	54	Rel. speed	w	m/s	12,7	17,2	22,3	27,6	33,1	38,6
55	55	Tang. force	F_x	N/m	69,0	77,9	80,4	80,6	79,2	74,8
56	56	Axial force	F_y	N/m	110,6	185,6	256,6	324,2	387,5	440,7
57	57									
58	58	Power	P_r	W	1118	2102	3037	3915	4705	5248
59	59	Swept area	A_i	m2	3,682	6,136	8,590	11,045	13,499	15,953
60	60		A_i	%	5	8	11	14	17	20
61	61	Axial force	F_yi	N	207,3	348,1	481,1	607,8	726,6	826,3
62	62	Table 4: Calculation								
63	63									
64	64	Pitch control	dbeta	°	0					
65	65	Wind speed	v_1	m/s	10					
66	66	Rotational speed	n	min^-1	88					
67	67	Power	P	kW	24,85					
68	68	Efficiency	eta_r	%	87,17					
69	69	Torque	M	Nm	2,697					
70	70	Axial force	T	N	4039					
71	71	Tip speed ratio	X_act	-	4,61					
72	72	Mean angle of attack	alpha_m	°	8,1					
73	73	Table 5: Summary of results								

Figure 7.2: Calculation of power and axial force (Formulas, see App. B)

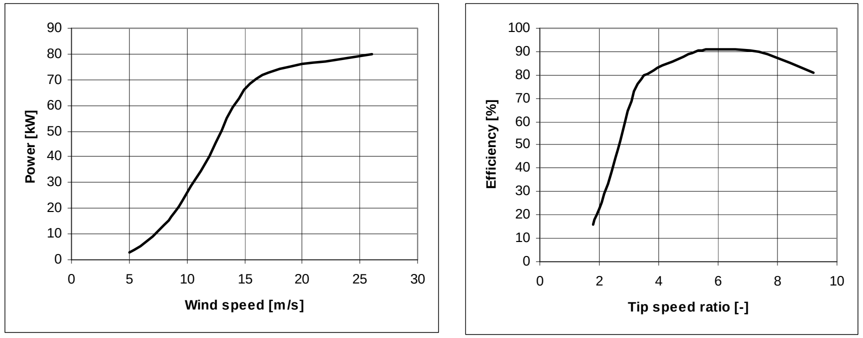


Figure 7.3: Power as function of wind speed (left) and efficiency as function of tip speed ratio (right)

8. Distribution of wind and annual energy production

Weibull distribution

The wind is distributed close to the Weibull distribution curve. For practical purposes one can calculate the probability for the wind being in the interval $v_i < v < v_{i+1}$

$$p(v_i < v < v_{i+1}) = \exp\left(-\left[\left(\frac{v_i}{A}\right)^k\right]\right) - \exp\left(-\left[\left(\frac{v_{i+1}}{A}\right)^k\right]\right) \text{ [-]} \qquad (8.1)$$

where A and k are found for a given site on the basis of measurements.

Annual production

If the power for the turbine at a given wind speed is $P(u)$, the annual production can be calculated as

$$E_{ann} = \sum \left\{8766\text{h} \cdot p(v_i < v < v_{i+1})\, P(v_m)\right\} \qquad \text{[J]} \qquad (8.2)$$

where v_m is the mean value of v_i and v_{i+1} i.e. $v_m = (v_i + v_{i+1})/2$.

Example

Typical values of A and k could be A = 8 m/s and k = 2. This will give the following distribution

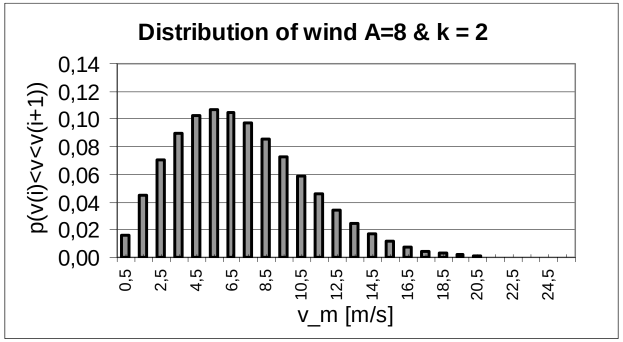


Figure 8.1: Distribution of the natural wind – for Δv = 1 m/s

Production curve: Let us imagine a power curve for a given stall controlled wind turbine given by

$$P(v) = \min(k_P v^3, P_N) \qquad \text{[W]} \qquad (8.3)$$

with k_P = 0,1 kW/(m/s)^3 and P_N = 200 kW.

The nominal power 200 kW would be reached at a wind speed of 12,6 m/s.

This would result in a power curve as given by figure 8.2.

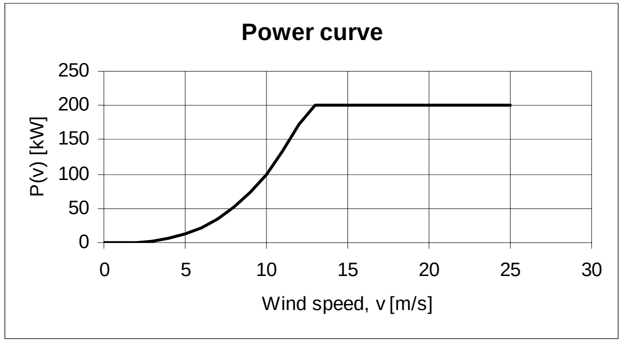


Figure 8.2. Power curve

If we combine the data from figure 8.1 and 8.2, we will get the energy production curve as shown in figure 8.3.

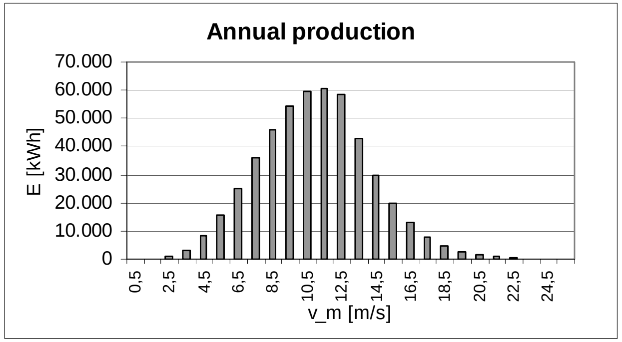


Figure 8.3: Annual distribution

What would be an optimal maximum power, P_{max} ? From figure 8.1 we can see that wind speed above 15 – 20 m/s are very rare. On the contrary, the power production of af wind turbine rises with a power of 3.

Figure 8.4 shows a calculation of the annual production as a function of the maximum power, P_{max}.

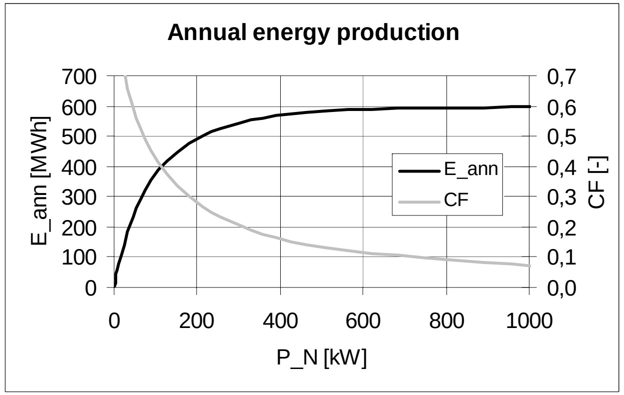


Figure 8.4: Annual energy production and capacity factor as function of nominal power

The capacity factor is definded as $CF = E_{ann} / (P_N \ 8766h)$.

Conclusion:
To answer the question we must know the price of the turbine, including tower and foundations, but more than about 200-300 kW does not seems reasonable.

9. Symbols

a	-	Axial interference factor
a'	-	Tangential interference factor
A	m/s	Wind speed, distribution curve
A	m^2	Area, swept area of the rotor
A_1	-	Constant
A_2	-	Constant
B_1	-	Constant
B_2	-	Constant
B	-	Number of blades
C_D	-	Coefficient of drag
$C_{D,max}$	-	Coefficient of drag, max value, at $\alpha = 90°$
C_{Dst}	-	Coefficient of drag, where stall begins
C_L	-	Coefficient of lift
C_{Lst}	-	Coefficient of lift, where stall begins
C_P	-	Power production factor
C_F	-	Axial force factor
C_y	-	Coefficient of axial forces
C_x	-	Coefficient of tangential forces
c	m	Chord length
E_{ann}	Wh	Annually produced energy
F	-	Calculation value
F_L	N/m	Lift force (per length of blade)
F_D	N/m	Drag force (per length of blade)
k	-	Constant
K	-	Factor
M	Nm	Torque
n	1/s	Rotational speed of rotor
p	Pa	Pressure
p_{tot}	Pa	Total pressure (Bernouilli's equation)
P	W	Power
P_N	W	Power, nominal wind
P_{max}	W	Max power of a given turbine
Q_M^*	N/m	Torque per length of blade
r	m	Radius to annular blade section (BEM theory)
Re	-	Reynold's number
T	N	Axial force (thrust) on the rotor
T^*	N	Axial force per width of the blades
U	N	Tangential force on the rotor
U^*	N	Tangential force per width of the blades
$u_2 = u$	m/s	Tangential wind speed in the rotor plane
$v_2 = v$	m/s	Axial wind speed in the rotor plane
v_1	m/s	Wind speed, upstream the rotor
v_3	m/s	Wind speed, down-stream the rotor
v_{TIP}	m/s	Tip speed of rotor blade
w	m/s	Relative wind speed
X	-	Tip speed ratio

x	-	Local speed ratio
α_A	°	Angle of attack, relative wind in relation to blade chord
α_{st}	°	Angle of attack, where stall begins
β	°	Pitch angle of the blade to rotor plane
γ	°	Relative wind to rotor axis
η	°	Efficiency
φ	°	Angle of relative wind to rotor plane
ω	s^{-1}	Angular velocity
μ	kg/(m s)	Dynamic viscosity
Δp	Pa	Differential pressure, over the rotor
Δw	m/s	Change of relative wind speed
Δu	m/s	Change of tangential wind speed
Δv	m/s	Change of wind speed
ρ	kg/m^3	Density of air (here 1,225 kg/m^3)

10. Literature

/1/ Andersen, P. S. et al
Basismateriale for beregning af propelvindmøller
Forsøgsanlæg Risø, Risø-M-2153, Februar 1979

/2/ Guidelines for design of wind turbines
Wind Energy Department, Risø, 2002, 2nd edition
ISBN 87-550-2870-5

/3/ Abbott, I. H., Doenhoff, A. E.
Theory of wing sections
Dover Publications, Inc., New York, 1959

/4/ Gasch, R; Twele, J.
Wind power plants - Fundamentals, Design, Construction and Operation
James and James, October 2005

App. A: Conservation of momentum and angular momentum

Momentum

Momentum of a particle in a given direction is defined as

$$p = m\,u \tag{A1}$$

where m is mass and u is speed of the particle

According to the Newton's 2nd law we have

$$F = \frac{dp}{dt} \tag{A2}$$

where F is the force acting on the particle

If the mass is constant, we have (Newton's 2nd law)

$$F = m\frac{du}{dt} = m\,a \tag{A3}$$

where a is the acceleration of the particle

If we have a flow of particles with the mass flow q_m we can calculate the force to change the velocity for u_1 to u_2 as

$$F = q_m\left(u_2 - u_1\right) \tag{A4}$$

Force equals differential pressure, Δp, times area, A, i.e. (A4) can be written as

$$\Delta p = \frac{q_m\left(u_2 - u_1\right)}{A} \tag{A5}$$

Example

For a wind turbine we have a wind speed up-stream the turbine of u_1 = 8 m/s and a wind speed down stream of u_2 = 2,28 m/s. In the rotor plane the wind speed is just the mean value of these to values, i.e. u = 5,14 m/s. The blade length is R = 25 m. Find the axial force on the rotor and the differential pressure over the rotor.

First we calculate the mass flow as

$$q_m = q_V\,\rho = \pi\,R^2\,u\,\rho = \pi\cdot 25^2\cdot 5{,}14\cdot 1{,}225 = 12369 \text{ kg/s}$$

Using (A4) we get F = 12369 (2,28 – 8,0) = -70,7 kN. The negative sign tells us that the force is in the opposite direction to the flow. The differential pressure is calculated by (5) giving Δp = 36 Pa.

Angular momentum:

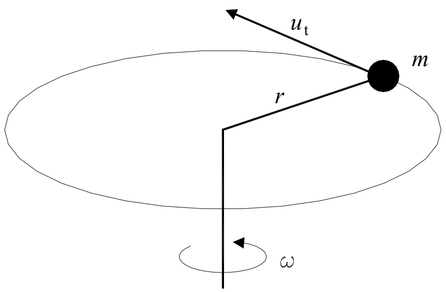


Figure A1: Rotating mass

Figure A1 shows a particle of mass m rotating a radius r with a tangential velocity of u_t. The angular momentum, L, is given by

$$L = m\,r\,u_t = m\,r^2 \frac{u_t}{r} = m\,r^2\,\omega = I\,\omega \qquad \text{(A6)}$$

where I is the moment of inertia and ω is the angular velocity.

The torque of the particle is given by

$$M = \frac{\mathrm{d}L}{\mathrm{d}t} = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\,\alpha \qquad \text{(A7)}$$

where α is the angular acceleration

Now consider a particle moving in a curved path, so that in time t it moves from a position at which it has an angular velocity ω_1 at radius r_1 to a position in which the corresponding values are ω_2 and r_2. To make this change we must first apply a torque, M_1, to reduce the particle's original angular momentum to zero, and then apply a torque, M_2, in the opposite direction to produce the angular momentum required in the second position, i.e.

$$M_1 = m\,r_1^2 \frac{\omega_1}{t} \qquad \text{(A8)}$$

and

$$M_2 = m\,r_2^2 \frac{\omega_2}{t} \qquad \text{(A9)}$$

The torque to produce the change of angular momentum can then be calculated as

$$M = M_2 - M_1 = \frac{m}{t}\left(\omega_2 r_2^2 - \omega_1 r_1^2\right) \qquad \text{(A10)}$$

This formula applies equally to a stream of fluid moving in a curved path, since m/t is the mass flowing per unit of time, q_m. Thus the torque which must be acting on a fluid will be

$$M = q_m\left(\omega_2 r_2^2 - \omega_1 r_1^2\right) \qquad \text{(A11)}$$

or

$$M = q_m\left(u_{2t} r_2 - u_{1t} r_1\right) \qquad \text{(A12)}$$

Example

Figure 2 shows a wind turbine with 2 blades. The blade length is R = 25 m and the rotational speed is n = 25 rpm which gives an angular velocity of ω = 2,62 s^{-1}.

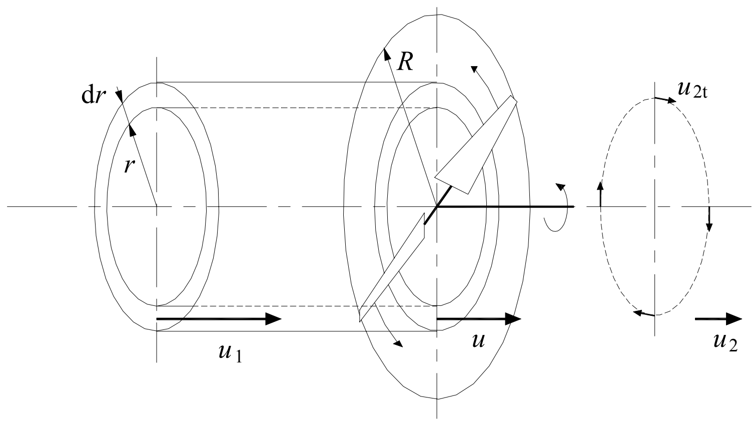


Figure A2: Wind turbine with 2 blades

Let us calculate the power for the annular element given by radius r = 17 m and with a thickness of dr = 10 cm. In a calculation concerning the BEM theory, one can find the axial velocity in the rotor plane at u = 5,14 m/s (a = 0,357) and at tangential velocity of the air after pasing the rotor plane at u_{2t} = 0,65 m/s (a' = 0,0072)

The mass flow through the annular element is

$$q_m = q_V \rho = \pi\left((r+\mathrm{d}r)^2 - r^2\right)u\,\rho = \pi\left((17+0{,}1)^2 - 17^2\right)\cdot 5{,}14\cdot 1{,}225 = 68{,}0 \text{ kg/s}$$

In formula (A12) we have $u_{1t} = 0$ because there is no rotation of the air before the rotor plane and $u_{2t} = 0{,}65$ m/s and $r_1 = r_2 = r = 17$ m. The torque can be calculated at

$$M = 68{,}0\left(0{,}65\cdot 17{,}0 - 0\cdot 0\right) = 755 \text{ Nm}$$

The power can be calculated at $P = M\,\omega = 1{,}98$ kW

App. B: Formulas, spread sheet calculations

Formulas in spread sheet, see figure 7.1

	E
14	=1,5/8
15	=E14*$E$7
16	=2/3*ATAN(1/$E$7/E14)*180/PI()
17	=E16-$E$9
18	=1/$E$8*16*PI()*E14/$E$10*(SIN(1/3*ATAN(1/$E$7/E14)))^2

Formulas in spread sheet, see figure 7.2

	E
22	=Design!E14
23	=E22*$E$6
24	=Design!E17
25	=Design!E18*$E$6
	E
29	**24,2317401773297**
30	**1,29343334743683**
31	=E29+$E$64
32	=MIN(E30*$E$10/(2*PI()*E23);1)
33	=E23*$E$11
	E
37	**0,316464512669469**
38	**0,24291710357473**
39	=ATAN((1-E37)/(1+E38)*$E$7/(E23*$E$11))/PI()*180
40	=E39-E31
41	=IF(E40<$J$11;$J$5+$J$6*E40+$J$7*E40^2+$J$8*E40^3+$J$9*E40^4;$J$16*SIN(2*E40*PI()/180)+$J$17*(COS(E40*PI()/180))^2/SIN(E40*PI()/180))
42	=IF(E40<=$J$11;$K$5+$K$6*E40+$K$7*E40^2+$K$8*E40^3+$K$9*E40^4;$J$14*SIN(E40*PI()/180)^2+$J$15*COS(E40*PI()/180)+$J$13)
43	=E41*COS(PI()/180*E39)+E42*SIN(PI()/180*E39)
44	=E41*SIN(PI()/180*E39)-E42*COS(PI()/180*E39)
45	=2/PI()*ACOS(EXP(-$E$10/2*($E$6-E23)/E23/SIN(E39*PI()/180)))
46	=4*E45*SIN(E39*PI()/180)^2/E32/E43
47	=1/(E46+1)
48	=1/2*(2+E46*(1-2*0,2)-SQRT((E46*(1-2*0,2)+2)^2+4*(E46*0,2^2-1)))
49	=IF(E37>0,2;E48;E47)
50	=1/(4*E45*SIN(PI()/180*E39)*COS(PI()/180*E39)/E32/E44-1)
51	=(E49-E37)/E37*100
52	=(E50-E38)/E38*100
53	
54	=$E$7*(1-E37)/SIN(PI()/180*E39)
55	=0,5*$E$9*E54^2*E30*E44
56	=0,5*$E$9*E54^2*E30*E43
57	
58	=$E$11*$E$10*$E$12*E55*E23
59	=PI()*(((E21+1)*$E$13)^2-(E21*$E$13)^2)
60	=E59/$E$14*100
61	=$E$10*E56*$E$13
	E
64	**0**
65	**10**
66	**88**
67	=SUM(E58:K58)/1000
68	=E67*1000/E15*100
69	=E67/E11
70	=SUM(E61:K61)
71	=E17
72	=AVERAGE(E40:K40)

The iteration to find *a* and *a'* in row 37 and row 38 (see example on figure 10) can be quite tedious. The job can be done automatically by the following macro. Run the macro by typing "Ctrl + p".

```
Sub Makro1()
'
' Makro1 Makro
' Makro indspillet 13-10-03 af Søren Gundtoft
'
' Genvejstast:Ctrl+p
'
    Range("E50:K50").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E38").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E49:K49").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E37").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E50:K50").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E38").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E49:K49").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E37").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E50:K50").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E38").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E49:K49").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E37").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E50:K50").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E38").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
    Range("E49:K49").Select
    Application.CutCopyMode = False
    Selection.Copy
    Range("E37").Select
    Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
        False, Transpose:=False
End Sub
```


App. C: Formulas, EES-programme

```
"BEM-model
2008.07.07/SGt"


PROCEDURE C_lift_and_drag(alpha:C_lift;C_drag)
"######################################################################"
"This procedure calculates the coefficient of lift and drag for the NACA23012 profile"
alpha_s=16
C_dmax=1
"lift and drag at stall"
C_lift_s=1,0318E-01+1,0516E-01*alpha_s+1,0483E-03*alpha_s^2+7,3487E-06*alpha_s^3-6,5827E-
06*alpha_s^4
C_drag_s=6,0387E-03-3,6282E-04*alpha_s+5,4269E-05*alpha_s^2+6,5341E-06*alpha_s^3-2,8045E-
07*alpha_s^4
"Some constants"
B1=C_dmax
B2=1/COS(alpha_s)*(C_drag_s-C_dmax*SIN(alpha_s)^2)
A1=B1/2
A2=(C_lift_s-C_dmax*SIN(alpha_s)*COS(alpha_s))*(SIN(alpha_s)/(COS(alpha_s))^2)
IF alpha<alpha_s THEN
C_lift=1,0318E-01+1,0516E-01*alpha+1,0483E-03*alpha^2+7,3487E-06*alpha^3-6,5827E-06*alpha^4
C_drag=6,0387E-03-3,6282E-04*alpha+5,4269E-05*alpha^2+6,5341E-06*alpha^3-2,8045E-07*alpha^4
ELSE
C_lift = A1*sin(2*alpha)+A2*(cos(alpha))^2/sin(alpha)
C_drag = B1*(sin(alpha))^2 + B2*cos(alpha) + C_drag_s
ENDIF
END

PROCEDURE interference_factors(V_0;omega;B;R;ri;c;betai: a_axial;a_tangential;C_x;C_y)
"######################################################################"

"Initialize"
a_axial=0,3
a_tangential=0,000001

REPEAT
        "Reset old iteration values"
        a_axial_old=a_axial
        a_tangential_old=a_tangential

        "Calculate angle of attack and solidity"
        phi=arctan(((1-a_axial)/(1+a_tangential))*(V_0/(ri*omega)))  {Determine rel. angle of attack, [rad]}
        alpha=phi-betai                      {Determine actual angle of attack correcting for pitch, [°] }
        sigma=(c*B)/(2*pi*ri)                {Determine solidity, [-] }
        CALL C_lift_and_drag(alpha:C_l;C_d)
        C_y=C_l*COS(phi)+C_d*SIN(phi)
        C_x=C_l*SIN(phi)-C_d*COS(phi)
        "Perform Prandtl tiploss correction"
        g=(B/2)*(R-ri)/(ri*sin(phi))
        expmg=exp(-g)
        F=(2/pi)*(arccos(expmg)*pi/180)
        K=4*F*SIN(phi)^2/sigma/C_y
        a_1=1/(4*F*SIN(phi)^2/sigma/C_y+1)
        hj=max((K*(1-2*0,2)+2)^2+4*(K*0,2^2-1);0,000001)
        a_2=1/2*(2+K*(1-2*0,2)-SQRT(hj))
        IF a_axial_old>0,2 THEN
        a_axialx=a_2
        ELSE
        a_axialx=a_1
        ENDIF
        a_tangentialx=1/(4*F*SIN(phi)*COS(phi)/sigma/C_x-1)
        "Underdamping"
        underdamping_factor=0,5
        a_axial=a_axial_old+underdamping_factor*(a_axialx-a_axial_old)
```

```
        a_tangential=a_tangential_old+0,5*(a_tangentialx-a_tangential_old)

        error_a=sqrt(( (a_axial-a_axial_old)/a_axial_old*100  )^2)
        error_t=sqrt(((a_tangential-a_tangential_old)/a_tangential_old*100)^2)
        epsilon=0,01
UNTIL((error_a<epsilon) AND (error_t<epsilon))

END "Procedure"

"###################################################################"
"                         M A I N   P R O G R A M                   "
"###################################################################"

R=5                 "[m]    Radius of rotor"
B=3                 "[-]    Number of blades"
V_0=10              "[m/s]  Wind speed"
n=88/60             "[rps]  Rotor speed"
rho_a=1,225         "[kg/m3] Density of air"
beta_p=0            "[°]    Pitch angle"
n_elements=8 "[-]   Number of ring elements"

omega = 2*pi*n      "[s^-1]       Angular speed"

"The model devides the blande in 8 ring element, with no air foil in the inner element"
DUPLICATE i=1;7
r_div_R[i]=(i+0,5)/8
END

"chord and pitch angle"
c_div_R[1]=0,2889  : beta_0[1]=23,92
c_div_R[2]=0,2249  : beta_0[2]=14,98
c_div_R[3]=0,1787  : beta_0[3]=9,96
c_div_R[4]=0,1474  : beta_0[4]=6,76
c_div_R[5]=0,1252  : beta_0[5]=4,56
c_div_R[6]=0,1088  : beta_0[6]=2,96
c_div_R[7]=0,0961  : beta_0[7]=1,74

DUPLICATE i=1;7
beta[i]= beta_0[i]+beta_p
END

"BEM"
DUPLICATE i=1;7
        r[i]=r_div_R[i]*R
        c[i]=c_div_R[i]*R
        CALL interference_factors(V_0;omega;B;R;r[i];c[i];beta[i]: a_axial[i];a_tangential[i];C_x[i];C_y[i])
        w_rel[i]=sqrt((omega*r[i]*(1+a_tangential[i]))^2+((1-a_axial[i])*V_0)^2)
        F_x[i]=0,5*rho_a*w_rel[i]^2*c[i]*C_x[i]
        F_y[i]=0,5*rho_a*w_rel[i]^2*c[i]*C_y[i]
        P[i]=R/8*B*omega*F_x[i]*r[i]
        Fa[i]=B*R/8*F_y[i]
END

P_rotor=SUM(P[i];i=1;7)/1000
F_rotor=SUM(Fa[i];i=1;7)

P_r_max=16/27*pi*R^2*1/2*rho_a*V_0^3/1000
eta_r=P_rotor/P_r_max*100
```


App. D: Formulas, EES-programme, to solve the integral in (6.4)

```
"Efficiency of a wind turbine rotor"
"08.07.2008/SGt"


FUNCTION fCp_schmitz(X)
"This function calculates the Cp-wake factor
according to Schmitz's theory
09.07.2008/SG"
i_max = 1000
dRR = 1/i_max
i = -1
x_sum = 0
REPEAT
i = i+1
RR = i/i_max
IF (RR=0) THEN
phi = 90
ELSE
phi = arctan(1/(X*RR))
ENDIF
xx = 4*X*RR^2*(sin(2/3*phi))^3/(sin(phi)^2)*dRR
x_sum = x_sum + xx
UNTIL (i=i_max)
fCp_schmitz = x_sum
END


"========= Main ========="

GR = 100        "[-] - Glide ratio = C_L/C_D"
B = 3           "[-] - Number of blades"
X=7             "[-] - Tip speed ratio"

Cp_Betz = 16/27
Cp_Schmitz = fCp_Schmitz(X)
eta_profile = 1-X/GR
eta_tip = (1-0,92/(B*sqrt(X^2+4/9)))^2
eta_tip_s = 1 - 1,84/B/X
Cp_Real = Cp_Schmitz*eta_profile*eta_tip
eta_wake = Cp_Schmitz/Cp_Betz
eta_rotor = eta_wake*eta_profile*eta_tip
```